Exact results for line operators in 4 and 3 dimensions

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Based on 1105.2568, 1111.4221, 1211.3409, 130x.xxxx
Plan

- Review of 4d story
- Vortex loop operators and mirror symmetry
- Localization computation
- Vortex operators from 't Hooft operators
- Conclusions

cf. [Kapustin, Willett, Yaakov]
4d $N=2$ gauge theories

- Eight supercharges
- Vector and hypermultiplets
- S-duality for $N=2$ superconformal theories with mass deformations (ex: $N=2^*$)
Line operators

- **Wilson operator**  \( W = \text{Tr} Pe^f A \)
  - Heavy electrically charged particle
  - Area law in a confining phase

- **'t Hooft operator**  \( \langle T \rangle = \int D\!A e^{-S} \)
  - Heavy(=singular) magnetic monopole
  - Area law in a Higgs phase
  - Disorder operator
Line operators

\[ \langle T \rangle = \int F = \frac{B}{2} \text{vol}(S^2) \]

Dirac quantization of \( B \)

Magnetic charge \( B = \text{coweight of } G \)

\( \Lambda_{\text{root}} \subset \Lambda_{\text{weight}} \subset \mathfrak{t}^* (G) = \text{dual Cartan} \)

\( \Lambda_{\text{coroot}} \subset \Lambda_{\text{coweight}} \subset \mathfrak{t} (G) = \text{Cartan} \)
Line operators

- Stronger constraint on B from matter.

- Can be supersymmetrized by coupling and exciting scalars. [Maldacena] [Kapustin]

- In localization, Wilson loop vevs are given by just inserting $\text{Tr}_R e^{2\pi a}$ into the matrix model.
S-duality

- Electric-magnetic duality (supersymmetric and non-Abelian)
- Corresponds to a change of pants decomposition of the Riemann surface
- Exchanges Wilson and 't Hooft operators
AGT correspondence

- N=2 gauge theories labeled by Riemann surfaces correspond to Liouville/Toda CFTs on the surfaces. [Alday, Gaiotto, Tachikawa]

- $Z_{\text{inst}} = \text{Conformal block}$

- $Z(S^4) = \text{Correlation function}$

- North and south pole contributions
AGT with line operators

Charges of line operators in 4d
\[\Longleftrightarrow\] Closed curves on the Riemann surface
[Drukker, Morrison, TO]

Line operators in 4d correspond to difference operators (Verlinde operators) acting on the conformal blocks.
[Drukker, Gomis, TO, Teschner] [Alday, Gaiotto, Gukov, Tachikawa, Verlinde] [Passerini] [Gomis, Le Floch]

Test involving Wilson loops was immediate, given Pestun’s localization computation on $S^4$. 
Results of localization

'\text{'t} Hooft operator expectation values on $S^4$ were computed. [Gomis, TO, Pestun]

$$\langle T(B) \rangle = \sum_v \int d\alpha \, Z_{1\text{-loop}}^{eq}(\alpha, v) Z_{\text{mono}}(\alpha, B, v)$$

$$\times \left| Z_{\text{cl}} \left( \alpha + i \frac{v}{2}, q \right) Z_{1\text{-loop}}^{\text{pole}} \left( \alpha + i \frac{v}{2} \right) Z_{\text{inst}} \left( \alpha + i \frac{v}{2}, q \right) \right|^2$$

$v$: coweights associated to $B$
Localization results for 't Hooft loops are consistent with S-duality and AGT correspondence.

Related results for $S^1 \times \mathbb{R}^3$ [Ito, TO, Taki] and $S^1 \times S^3$ [Gang, Koh, Lee].
3d N=2 theories

- Field content = dimensional reduction of 4d N=1 theory.
- Can have a Chern-Simons term.
- Will consider abelian gauge groups only.
Mirror symmetry = particle/vortex duality
= change of triangulation of a 3-manifold
(and a polarization if \( \exists \) boundary) in the 3d/3d correspondence

N=2 SUSY gauge theories have 1/2 BPS Wilson operators.
Gauge theories in 3d can be mirror to theories without gauge fields.

What is the mirror dual of a Wilson operator?
Vortex line operators

A vortex line operator is a disorder operator characterized by a vortex-like singularity in the background gauge field

\[ A \sim \alpha d\varphi \]

Was studied in ABJM using holography. [Drukker, Gomis & Young]

1/2 BPS in N=2 SUSY.

Gauge field may be non-dynamical

\[ \Rightarrow \text{Flavor vortex operator} \]
Chasing line operators through dualities

Assume the gauge group is abelian.

Mirror map for global symmetries determines how loop operators transform for the following reasons.
Wilson loop as a flavor vortex loop

\( J = \ast dA \) is a conserved current for topological symmetry \( U(1)_J \).

\[ \int a_\mu J^\mu = \int a \wedge dA = \int da \wedge A \]

Flavor vortex loop for \( U(1)_J \)

\[ da = \alpha \delta(\text{loop}) \]

\( \Rightarrow \) Wilson loop

\[ \exp \left( -i \alpha \oint A \right) \]

Can be supersymmetrized.
Loop operators associated with global symmetries

- Each global symmetry (commuting with SUSY) has associated loop operators.
- Wilson loops are associated with $U(1)_J$ generated by $J=\star dA$.
- Flavor vortex loops are associated with flavor symmetries.
- Mirror symmetry may map $U(1)_J$ to an ordinary flavor symmetry.
Charge (non)quantization

Topology of spacetime, via gauge invariance, determines the domain of $\alpha$:

$$\alpha \in \mathbb{Z} \quad \text{or} \quad \alpha \in \mathbb{R}$$

$S^3$ minus the loop is topologically $S^1 \times D^2$. Arbitrary holonomy $\alpha$ is allowed. Also on $S^3$, there is no gauge transformation under which $\exp \left( -i \alpha \oint A \right)$ is not invariant for $\alpha \in \mathbb{R}$. 
Charge (non)quantization

*On $S^1 \times S^2$, single vortex and Wilson loops along $S^1$ must have integer charges. Two such loops at the north and south poles of $S^2$ must have the total charge quantized.*
Trivial loop operators

- Wilson loops made of a non-dynamical gauge field (flavor Wilson loops) act as multiplication by an overall factor.

- Vortex loops for a dynamical gauge field turn out to be equivalent to flavor Wilson loops for topological symmetry $U(1)_J$. 
Mirror symmetry action on loop operators

- Topological symmetry $U(1)_J$ in one theory is mapped to ordinary flavor symmetry that rotates chirals.
- Gauge Wilson loops are mapped to flavor vortex loops.
- Gauge vortex loops are mapped to flavor Wilson loops. Both are trivial.
- Brane and surface operator constructions confirm this dictionary.
Example

- $N=2$ SQED with two chirals with charges +1 and -1.

- $N=2$ theory with three chirals $X, Y, Z$ interacting via a superpotential $W=XYZ$. 
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<td>$U(1)_J$</td>
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<td>Wilson loop</td>
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<td>$U(1)_J$</td>
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Localization calculation

For $N=2$ SUSY theories on the ellipsoid $S^3_b$ and $S^1 \times S^2$, the partition functions are known.

The one-loop determinant was computed by spherical harmonics. [Kapustin-Willett] [Hama, Hosomichi &Lee] [Kim] [Imamura&Yokoyama]. We do this using the equivariant index theorem.
Deform action by $tQ \cdot V$ for appropriate $V$. Take $t \to \infty$.

In Pestun’s approach to localization, the one-loop determinant is obtained from the equivariant index (weighted trace of $Q^2$=bosonic symmetry) of a differential operator in $V$.

$$\text{ind } D = \sum_{\alpha} c_{\alpha} e^{w_{\alpha}} \to Z_{1\text{-loop}} = \prod_{\alpha} w_{\alpha}^{-c_{\alpha}}$$

On $S^4$ and $S^2$, the index and the one-loop determinants receive contributions from the fixed points (north and south poles).
Localization on $S^3_b$ and $S^1 \times S^2$ via the index theorem

On $S^3_b$ and $S^1 \times S^2$, there is no fixed point for the relevant isometries. Is the index zero?

When one of the isometries acts on the geometry without a fixed point, the (transversally elliptic) differential operator reduces to a (transversally elliptic) operator on the quotient space ($S^2$ in our case).
The other isometry has two fixed points on $S^2$ (two circles in 3-manifolds) which contribute to the one-loop determinant.

(May be possible to explain the observed factorization of the partition function on $S^3_b$ and $S^1 \times S^2$ in the literature. [Pasquetti])
Example: tetrahedron theory and its mirror

- Theory of a single chiral multiplet coupled to a background vector multiplet via a level $-1/2$ Chern-Simons term. Set $\Delta=1$ for simplicity.

- A mirror dual is a $U(1)$ gauge theory at CS level $1/2$ coupled to a single chiral multiplet with $\Delta=0$. 
Partition functions of the tetrahedron theory

On the ellipsoid $S^3_b \ (b^2|z_1|^2 + b^{-2}|z_2|^2 = 1)$, with mass $m$, the partition function is

$$Z(S^3_b) = e^{\frac{\pi i}{2} m^2} s_b(-m) \quad s_b(x): \text{double sine function}$$

On $S^1 \times S^2$, the partition function (generalized index) with magnetic flux $n$ and flavor fugacity $\zeta$ is

$$Z(S^1 \times S^2) = \text{Tr}_{\mathcal{H}_n} (-1)^F q^{R/2 + j_3} \zeta^f = \prod_{r=0}^{\infty} \frac{1 - q^{r - \frac{n}{2} + 1/2} \zeta}{1 - q^{r - \frac{n}{2} + 1/2} \zeta - 1}$$
Localization with a vortex loop

Turning on the vortex loop $V_\alpha$ induces a spectral flow on the eigenmodes of the kinetic operator.

We allow some modes that are singular along the loop to fluctuate, as part of the definition of the operator.

The classical and one-loop contributions get modified.
The effect of a flavor vortex loop $V_\alpha$ on the partition function $Z(S^3_b)$ is to shift mass $m$:

$$m \rightarrow m - ib^{\pm 1} \alpha$$

The effect of $V_\alpha$ on $Z(S^1 \times S^2)$ is to shift flux $n$ and fugacity $\log \zeta$:

$$n \rightarrow n + \alpha, \quad \zeta \rightarrow \zeta q^{\pm \alpha/2}$$

Agree with localization results for Wilson loops in the mirror theory.
Vortex loops from \('t\) Hooft loops

- A flavor vortex loop with a quantized charge acts in the same way on a 3d partition function as a bulk \('t\) Hooft loop brought to the boundary.

- The relation can be intuitively derived.
Freshman electrostatics problem (up to S-duality)

Dirac monopole $\Rightarrow$ vortex singularity

'\textquoteleft t \textquoteright$ Hooft operator $\Rightarrow$ vortex operator
Conclusions

- S-dual of a Wilson loop in 4d is a 't Hooft loop.
- Mirror of a Wilson loop (for an abelian gauge group) in 3d is a flavor vortex loop.
Conclusions

- Mapping of loop operators under mirror symmetry can be tracked by chasing global symmetries that commute with SUSY.

- Mirror symmetry for loop operators can be confirmed by localization on $S^3_b$ and $S^1 \times S^2$.

- Exact localization calculations are possible for disorder operators such as 't Hooft and vortex loops. Results consistent with dualities and 2d/4d+3d/3d relations.
To be explored...

- Vortex loops and mirror symmetries for non-abelian symmetries.
- Analog of monopole bubbling seen for a 't Hooft loop in 4d?
- Line operators/boundaries in 2d theories.