Review of the deformation theory of prepotential and the S-W system

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on occasion of the workshop celebrating Prof. S. Aoyama’s happy retirement

0) largely chronological

• references taken from my past work:

  I-Morozov in 1995 IM1-3
  in 2002 IM4-7
  I-Kanno in 2003 IKan2
  Fujiwara-I-Sakaguchi 2004- FIS1
  I-Maruyoshi in 2007 I-Maruyoshi2,3
  I-Maruyoshi-Oota in 2009 IMO
  I-Oota in 2010 IO5

  S. Aoyama-Y. Kodama in 1995
  S. Aoyama-T. Masuda in 2004
  S. Aoyama in 2005
Contents

I) Part 1 1995-

II) Part 2 2002-

III) Part 3 2009-

warning: models hopping from one to the other

A tentative conclusion: \( \mathcal{F} = F_{\text{mat}} \)

Sorry. This talk is patchwork.
The curve for $\mathcal{N} = 2$, $SU(N)$ pure super Yang-Mills is a hyperelliptic Riemann surface of genus $N-1$:

$$Y^2 = P_N^2(x) - 4\Lambda^{2N}$$

$$P_N(x) = \langle \det(x1 - \Phi) \rangle \equiv \prod_{i=1}^{N} (x - p_i) = x^N - \sum_{k=2}^{N} u_k x^{N-k}$$

$$\ln \det = \text{tr} \ln \quad \rightarrow \quad \ln \det = \sum_{k=0}^{N} s_k(h_\ell)x^{N-k}$$

Here

$$h_\ell = \frac{1}{\ell} \langle \text{tr} \Phi^\ell \rangle = \frac{1}{\ell} \sum_{i=1}^{N} p_i^\ell$$

$$\ell = 2, \cdots, N$$

Moduli

Via the spectral parameter $z$

the curve is written as

$$P_N(x) = z + \frac{\Lambda^{2N}}{z}$$

$$Y = z - \frac{\Lambda^{2N}}{z}$$

N site periodic Toda chain
• The distinguished meromorphic differential for the prepotential theory is

\[ d\hat{S}_{SW} = xd \log z = xt(x)dx, \quad t(x) = \frac{P'_N}{\sqrt{P^2_N - 4\Lambda^2}} \]

\[ \exists \text{double pole at } \infty \leftrightarrow \text{only } T_1 \text{ turned on} \]

• The defining property:

moduli derivatives are holomorphic

\[ \left. \frac{\partial}{\partial u_k} d\hat{S}_{SW} \right|_{z,\Lambda} = \frac{x^{N-k}}{Y} dx \]

or

\[ \left. \frac{\partial}{\partial u_k} d\hat{S}_{SW} \right|_{x,\Lambda} = \frac{x^{N-k}}{Y} dx - d \left( \frac{x^{N-k+1}}{Y} \right) \]

Q1: Which one to fix?

Q2: What to regard as moduli?
• (Generic) prepotential theory: Whitham deformation
deform both moduli of RS and the mero. differential consistently
without losing the defining property:

\[ d\hat{S}_W \to d\hat{S}; \quad (\bullet) \quad \frac{\partial}{\partial h_k}d\hat{S}|_{*,\Lambda} = \text{holomorphic} \]

adding higher order poles to the original double pole:

\[ \xi: \text{local coord. at these} \quad \xi = z^{\frac{1}{N}} \text{ or } x^{-1} \]

Introduce \( d\Omega_\ell : \text{a set of mero. diff.} \)

\[ \text{s. t. } d\Omega_\ell = \xi^{-\ell-1}d\xi + \text{nonsingular} \quad \ell > 1 \]

In order to remove ambiguities, require

\[ (\ast) \quad \int_{A_i} d\Omega_\ell = 0 \quad \text{cf. } d\omega_i : \text{canonical hol. diff.} \]

\[ \text{s. t. } \int_{A_i} d\omega_j = \delta_{ij} \]

The ones without \((\ast)\) are denoted by \( d\hat{\Omega}_\ell \)
• **The upshot is**

\[
d\hat{S} = \sum_{i=1}^{g} a^i d\omega_i + \sum_{\ell \leq 1} T_\ell d\Omega_\ell \quad \Rightarrow \quad \text{regard } h_k = h_k(a^i, T_\ell)
\]

\[
T_\ell = \operatorname{res}_{\xi=0} \xi^\ell d\hat{S} \quad \Rightarrow \quad \text{time variables or T moduli}
\]

local coord. in the moduli space

• **Introduce the prepotential** \( \mathcal{F}(a^i, T_\ell) \) via

\[
(\star) \quad \frac{\partial \mathcal{F}}{\partial a^i} = \int_{B^i} d\hat{S}, \quad \frac{\partial \mathcal{F}}{\partial T_\ell} = \frac{1}{2\pi i \ell} \operatorname{res}_{\xi=0} \xi^{-\ell} d\hat{S} \equiv \mathcal{H}_{\ell+1}(h_k)
\]
Reasoning to

\[ \sum_{\ell \leq 1} T_\ell \partial \Omega_\ell \]

Introduce the time variables \( T_\ell \) via a soln \( \hat{S}(T_\ell|h) \) to (●)

\[ \frac{\partial \hat{S}}{\partial T_\ell} = \partial \Omega_\ell \quad (1) \quad \Rightarrow \quad \frac{\partial a_i}{\partial T_\ell} = 0 \]

\[ \frac{\partial}{\partial h_k} \partial \hat{\Omega}_\ell = \sum_{i=1}^{g} \sigma_{ki}^{(\ell)} d\omega_i \]

\[ \hat{\Omega}_\ell = \partial \Omega_\ell + \sum_{i=1}^{g} c_i^{(\ell)} d\omega_i \quad (2) \]

Expand the solutions as

\[ d\hat{S} = \sum_{m} \beta_m(T) d\hat{\Omega}_m(h) \quad (+) \]

\[ \frac{\partial d\hat{S}}{\partial T_n} = \sum_{m} \left( \frac{\partial \beta_m}{\partial T_n} d\hat{\Omega}_m + \beta_m \sum_{k} \frac{\partial h_k}{\partial T_n} \sum_{i=1}^{g} \sigma_{ki}^{(m)} d\omega_i \right) \quad (3) \]
\[(1),(2),(3) \implies \]
\[
\frac{\partial \beta_m}{\partial T_n} = \delta_{m,n} \quad \text{i.e.} \quad \beta_m(T) = T_m \quad \text{—— (4)}
\]
\[
\sum_k \frac{\partial h_k}{\partial T_n} \left( \sum_m T_m \sigma_{ki}^{(m)} \right) = -c_i^{(n)} \quad \text{—— (5)}
\]
\[(+) ,(4),(2) \implies \]
\[
d \tilde{S} = \sum_m T_m d\Omega_m + \sum_m T_m \sum_i c_i^{(m)} d\omega_i = a^i
\]

can check instead
\[
\frac{\partial}{\partial a_i} \left. d \tilde{S} \right|_{T_{\ell}} = d\omega_i, \quad a_i \left( h_k(T) \right)
\]
Picture we want to materialize as prepotential theory:

\[ \mathcal{N} = 2 \]

\( \text{U(N) pure SYM} \)

deformed by superptl

\[ g_{n+1} \int d^2 \theta \text{tr} W_{n+1}(\Phi) \] \hspace{2cm} \text{s.t. } W'_{n+1}(x) = \prod_{i=1}^{n} (x - \alpha_i) \]

\( \mathcal{N} = 1 \)

\[ \prod_{i=1}^{n} \times \text{U}(N_i) \]

\( \text{LEEA} \)

\( \text{U(1)}^{N-1} \times \text{U}(1) \)

deconfining

\( \text{Coulomb} \)

\( \text{Coulomb vacua is codimension subspace in } \mathcal{N} = 2 \text{ Coulomb branch} \)

\( \text{and is parametrized by the order parameters (gluino condensates)} \)

\[ S_i \propto \text{Tr}_{\text{SU}(N_i)} W^\alpha W_\alpha, \quad i = 1, \ldots, n \]
We impose

\[ \ker \frac{\partial^2 F}{\partial a^i \partial T_\ell} \neq 0, \quad \text{or} \quad \operatorname{rank} \frac{\partial^2 F}{\partial a^i \partial T_\ell} \leq N - 2 \]
• **Implications**

**i)** \( \exists \) a nonvanishing column vector

\[
0 = \sum_{\ell} \frac{\partial^2 F}{\partial a^i \partial T_{\ell}} c^\ell = \sum_{\ell} \int_{B_i} d\Omega_{\ell} c^\ell = \frac{1}{2\pi i} \text{res}_{\xi=0} \left( \sum_{\ell} \frac{c^\ell}{\ell} \xi^{-\ell} \right) d\omega_i
\]

\( \overset{(1)}{\Rightarrow} d\hat{\Omega} = \sum_{\ell} c^\ell d\Omega_{\ell} \) have vanishing periods over all \( A_i \) & \( B^i \) cycles.

\[\Rightarrow f(z) = \int z d\hat{\Omega}.\]

contradictory to the Weierstrass gap theorem (\( \iff \) Riemann-Roch Th.)

To avoid \( \times \), degeneration need

**ii)** \( \exists \) a nonvanishing row vector \((\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_{N-1})\)

\[
0 = \sum_{i=1}^{N-1} \tilde{c}_i \frac{\partial^2 F}{\partial a^i \partial T_{\ell}} = \sum_{i} \tilde{c}_i \frac{\partial h_{\ell+1}}{\partial a^i} \iff \sum_{i} \tilde{c}_i \frac{\partial h_{\ell+1}}{\partial a^i} = 0
\]

\[\therefore \text{ moduli depend actually on less than } N-1 \text{ arguments}\]
The Weierstrass “gap” Theorem

\[ M, g, P \in M, \quad \exists g \text{ integers} \]

\[ 1 = n_1 < n_2 < \cdots < n_g < 2g \]

s. t.

there does NOT exist a fn \( f \) holo. on \( M \setminus \{P\} \) with a pole of order \( n_j \) at \( P \).
• Factorization/degeneration & the matrix model curve

Let \( n-1 \) be \( \#(g) \) after degeneration

\[
\begin{align*}
Y^2 &= H_{N-1}(x)^2 F_{2n}(x) \\
P_N'(x) &= H_{N-n}(x) R_{n-1}(x)
\end{align*}
\]

\[ t(x) = \frac{R_{n-1}(x)}{\sqrt{F_{2n}(x)}} \]

and

Finally examine (2)

\[
\sum_{\ell=1}^{N-1} \frac{c_{\ell}}{\ell} (\xi^{\ell})_+ \equiv W_{k+1}(x) \equiv \prod_{j=1}^{k} (x - \alpha_j)
\]

poly. of degree \( k(\ge n) \) in \( x \).

serve as bases of holo. diff. of the reduced R.S.

\[
\begin{align*}
0 &= \text{res}_{x=\infty} \left( W_{k+1}(x) \frac{x^{j-1}}{\sqrt{F_{2n}}} \right) \\
\therefore \quad \frac{W_{k+1}'}{\sqrt{F_{2n}}} &= Q_{k-n}(x) + \sum_{\ell>n} \frac{\beta_{\ell}}{x^\ell}
\end{align*}
\]

deg.

\[
\therefore \quad y^2 = F_{2n} Q_{k-n}^2 = W_{k+1}' + f_{k-1}
\]

This is the curve appearing in the n-cut solution of the matrix model.
• **Classical limit** \( \Lambda = 0 \) largely simplified

\[
Y = z = \prod_{\ell=1}^{N} (x - p_{\ell})
\]

\[
d\hat{S}_{SW}^{(\text{class})} = x \sum_{i=1}^{N} \frac{1}{x - p_i} \, dx
\]

\[
= \sum_{i=1}^{N} p_i d\omega_i + N \, dx
\]

\[
\therefore \quad a_i^{(\text{class})} = p_i
\]

\[
\therefore \quad d\omega_i^{(\text{class})} = \frac{dx}{x - p_i}
\]

• **Suppose**

\[
z = \prod_{j=1}^{n} (x - \beta_j)^{N_j}, \quad \sum_{j=1}^{n} N_j = N
\]

i.e. \( N_j \) poles coalesce at \( \beta_j, \quad j = 1, \cdots, n \)

\[
d\omega_j^{(\text{class}, \text{red})} = \frac{dx}{x - \beta_j}
\]
the condition (2)

\[ 0 = \lim_{x \to 0} \left( W'_{k+1}(x) d\omega^{\text{class,red}}_j \right), \quad j = 1, \cdots, n \]

(contour deformation)

\[ \beta_j \text{ must coincide with one of the root } \alpha_j \text{ of } W'_{k+1} \]

The value of \( \Phi \) constrained to the extremum of \( W'_{k+1} \)

\[ \therefore \quad W'_{k+1} \text{ tree level superpotential} \]
We have the reduced curve of \[ g = n - 1 \]

\[ y^2 = W'_{n+1}(x; \alpha_j)^2 + f_{n-1}(x) \]

denote the coeff by \( b_\ell(\alpha_j) \) & temporarily forget the \( \alpha \) dependence

\[ \dim(\text{moduli}) = 2n \]

\( \approx \) cut lengths + cut position

**What should \( d\hat{S}_{\text{mat}} \) be?**

demand almost holomorphic after \( b_\ell \) derivatives

cf. bases \( \frac{x^{j-1}}{y}, \ j = 1, \cdots, n-1, n \)

obviously \( d\hat{S}_{\text{mat}} = y(x)dx \), \( T_1, T_2, \cdots, T_n, T_{n+1} \) turned on but no reason to set

\[ S_i = \int_{A_i} d\hat{S}_{\text{mat}} \ i = 1, \cdots, n \]

and

\[ \frac{\partial F}{\partial S_i} = 2 \int_{\text{cutoff}} d\hat{S}_{\text{mat}} \]

\[ S = \sum_{i=1}^{n} S_i = \int_{\prod_{i=1}^{n} \cup A_i} d\hat{S}_{\text{mat}} \]

equal to zero \( \Rightarrow \exists \) cutoff

originally \((\star)\) + small cut expansion \( \Rightarrow \) answer for \( F \) to order \( S_i^3 \)
• Calculus from T moduli
  Yet, ∃ simpler procedure thanks to Whitham machinery

\[
T_{m+1} = \text{res } x^{-m-1} d\hat{S}_\text{mat} = g u_m, \quad u_m = (-)^{n-m} e^{(\alpha)}_{n-m} \\
e_m(\alpha) = \sum_{i_1 < \cdots < i_m} \alpha_{i_1} \cdots \alpha_{i_m}
\]

\[
(\ast \ast) \quad \frac{1}{g} \frac{\partial F}{\partial u_\ell} = \frac{\partial F}{\partial T_{\ell+1}} = \frac{1}{\ell + 1} \text{res}(x^{\ell+1} - \Lambda^{\ell+1}) d\hat{S}_\text{mat}
\]

and parameterize

• \(d\hat{S}_\text{mat}\) expandable in \(\tilde{S}_i\)

• \(A_i\) cycle integrations & inversion give

\[
\tilde{S}_i = S_i + \frac{1}{2g} \sum_{j,k} \frac{1}{\alpha_{ij} \alpha_{ik} \Delta_i} S_j S_k + \cdots
\]

\[
\Rightarrow \text{R.H.S of } (\ast \ast) = \sum_i S_i \left( \frac{\partial W_{n+1}(\alpha_i)}{\partial u_\ell} - \frac{\partial W_{n+1}(\Lambda)}{\partial u_\ell} \right) - \frac{1}{4} \sum_{j<k} \left( S_j^2 + S_k^2 - 4S_j S_k \right) \frac{\partial}{\partial u_\ell} \log \alpha_{jk} + \cdots
\]

\(\Rightarrow \text{answer}\)
• Proposed form of $\mathcal{F}(S|\alpha)$:

$$2\pi i \mathcal{F}(S|\alpha) = 4\pi i g_{n+1} \left( W_{n+1}(\Lambda) \sum_i S_i - \sum_i W_{n+1}(\alpha_i) S_i \right) - \left( \sum_i S_i \right)^2 \log \Lambda +$$

$$+ \frac{1}{2} \sum_{i=1}^n S_i^2 \left( \log \frac{S_i}{4} - \frac{3}{2} \right) - \frac{1}{2} \sum_{i<j} (S_i^2 - 4S_i S_j + S_j^2) \log \alpha_{ij} + \sum_{k=1}^\infty \frac{\mathcal{F}_{k+2}(S|\alpha)}{(i\pi g_{n+1})^k}$$

$k + 2$ order poly. in $S_i$

• Our result:

$$\mathcal{F}_3(S|\alpha) = \sum_{i=1}^n u_i(\alpha) S_i^3 + \sum_{i\neq j} u_{i:j}(\alpha) S_i S_j + \sum_{i<j<k} u_{i:j:k}(\alpha) S_i S_j S_k,$$

$$\mathcal{F}_{k+2}(S|\alpha) = \sum_{i=1}^n u_i(\alpha) S_i^3 + \sum_{i\neq j} u_{i:j}(\alpha) S_i S_j + \sum_{i<j<k} u_{i:j:k}(\alpha) S_i S_j S_k,$$

$$u_i(\alpha) = \frac{1}{6} \left( - \sum_{j(\neq i)} \frac{1}{\alpha_{ij}^2 \Delta_j} + \frac{1}{4\Delta_i} \sum_{j,k(\neq i)} \frac{1}{\alpha_{ij} \alpha_{ik}} \right),$$

$$u_{i:j}(\alpha) = \frac{1}{4} \left( - \frac{3}{\alpha_{ij}^2 \Delta_i} + \frac{2}{\alpha_{ij}^2 \Delta_j} - \frac{2}{\alpha_{ij} \Delta_i} \sum_{k(\neq i,j)} \frac{1}{\alpha_{ik}} \right),$$

$$u_{i:j:k}(\alpha) = \frac{1}{\alpha_{ij} \alpha_{ik} \Delta_i} + \frac{1}{\alpha_{jj} \alpha_{jk} \Delta_j} + \frac{1}{\alpha_{kk} \alpha_{kj} \Delta_k},$$

$$\Delta_i = W''_{n+1}(\alpha_i) = \prod_{j(\neq i)} \alpha_{ij}$$

IM6

Aoyama for the case with fund. matter

cf. Shigemori et al. originally done by the small cut expansion
• Case of spontaneously broken $\mathcal{N} = 2$ supersymmetry

The action $S_{\mathcal{N}=2}^{\mathcal{F}}$

$$S_{\mathcal{N}=2}^{\mathcal{F}} = \int d^4x d^4\theta \left[ -\frac{i}{2} \text{Tr} \left( \Phi e^{a}v \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi} - h.c. \right) + \xi V^0 \right]$$

$$+ \left[ \int d^4x d^2\theta \left( -\frac{i}{4} \frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi_a \partial \Phi_b} \mathcal{W}^{a}_a \mathcal{W}^{b}_a + e\Phi^0 + m \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi^0} \right) + h.c. \right],$$

$\xi$, $e$, $m$; electric & magnetic F-I terms

large $(\xi, e, m)$ \quad \text{small} $(\xi, e, m)$

$I$-Maru, 2011

Prepotential function $\mathcal{F}$

single trace function of a polynomial in $\Phi$

$$\mathcal{F}(\Phi) = \sum_{\ell=1}^{n+1} \frac{g_{\ell}}{(\ell + 1)!} \text{Tr} \Phi^{\ell+1}, \quad \text{deg}\mathcal{F} = n + 2.$$

Matter induced part of the effective superpotential

$$e^{i \int d^4x (d^2\theta W_{eff} + h.c. + (D-term))} = \int D\Phi D\Phi e^{i S_{\mathcal{N}=1}^{\mathcal{F}}}.$$
• Generalized Konishi anomaly equations

The anomalous Ward identity of our model for the general transformation

\[ \delta \Phi = f(\Phi, \mathcal{W}) \]

is

\[ - \left\langle \frac{1}{64\pi^2} \left[ \mathcal{W}_\alpha, \left[ \mathcal{W}_\alpha, \frac{\partial f}{\partial \Phi_{ij}} \right] \right]_{ij} \right\rangle = \left\langle \text{Tr} f'W'(\Phi) \right\rangle - \left\langle \frac{i}{4} \text{Tr}(f''''(\Phi)W_\alpha W_\alpha) \right\rangle. \]

In terms of the two generating functions

\[ R(z) = -\frac{1}{64\pi^2} \left\langle \text{Tr} \frac{\mathcal{W}_\alpha \mathcal{W}_\alpha}{z - \Phi} \right\rangle, \]

\[ T(z) = \left\langle \text{Tr} \frac{1}{z - \Phi} \right\rangle, \]

\[ R(z)^2 = W'(z)R(z) + \frac{1}{4}f(z), \]

\[ 2R(z)T(z) = W'(z)T(z) + 16\pi^2 iF'''(z)R(z) + \frac{1}{4}c(z), \]

where \( f(z) \) and \( c(z) \) are polynomial of degree \( n-1 \) and

\[ F'''(z) = \sum_{\ell=2}^{n+1} \frac{g_\ell z^{\ell-2}}{(\ell - 2)!} = \frac{W''(z)}{m}. \]

The explicit forms of \( f(z) \) and \( c(z) \) are not needed.

The equation for \( R(z) \) is the same as that of CDSW which is identified with the loop equation of the matrix model. The equation for \( T(z) \) alters. This lead to the deformation of our effective superpotential from the well-known form in \( S_{N=1} \).
• Formula for the effective superpotential

define the one point functions as

\[ v_\ell = -\frac{1}{64\pi^2} \langle \text{Tr} \, \mathcal{W}^\alpha \mathcal{W}_\alpha \Phi^\ell \rangle, \quad u_\ell = \langle \text{Tr} \Phi^\ell \rangle, \quad \text{for } 1 \leq \ell \leq n + 1. \]

In terms of \( v_\ell \), we define \( F \) as

\[ \frac{\partial F}{\partial g_\ell} = \frac{m}{\ell!} v_\ell, \quad \text{for } 1 \leq \ell \leq n + 1. \]

Using \( F \),

\[ W_{\text{eff}} = \sum_i N_i \frac{\partial F}{\partial S_i} + \frac{16\pi^2 i}{m} \sum_{\ell=2}^{n+1} g_\ell \frac{\partial F}{\partial g_{\ell-1}}. \]

• Comments and outline of the proof

• \( R_m(z) = R(z) \), \hspace{1cm} \therefore \quad F = F_{\text{mat}}

• (\ast) equivalent to

\[ T(z) = \sum_i N_i \frac{\partial R(z)}{\partial S_i} + \frac{16\pi^2 i}{m} \sum_{\ell=2}^{n+1} g_\ell \frac{\partial R(z)}{\partial g_{\ell-1}}. \]
What is $Z_{\text{Nek}}^{\epsilon_1, \epsilon_2}$?

$\mathcal{F}_{SW}^{(\text{eff})}(a_i)$; LEEA of $\mathcal{N} = 2$ SU(N) SUSY gauge theory

$a_i = (\phi_i)$; undetermined VEV called Coulomb moduli

$q_{\text{bare}} = e^{\pi i \tau_{\text{bare}}}, \quad \tau_{\text{bare}} = \frac{\theta}{\pi} + \frac{8\pi i}{g_{\text{bare}}^2}$

$\mathcal{F}_{SW}^{(\text{eff})} = \mathcal{F}_{\text{1-loop}}^{(SW)} + \mathcal{F}_{\text{inst}}^{(SW)}$

$\mathcal{F}_{\text{instanton}}$; instanton contributions

Result of Nekrasov;

$\mathcal{F}_{\text{inst}}^{(SW)}$ is microscopically calculable in the presence of $\Omega$ background

$\sim$ deformation parameter $\epsilon_1, \epsilon_2$

as

$Z_{\text{Nek}}(\epsilon_1, \epsilon_2, a_i; q) = \exp\left(\frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}_{\text{inst}}(\epsilon_1, \epsilon_2, a_i)\right), \quad \mathcal{F}_{\text{inst}}(0, 0, a_i) = \mathcal{F}_{\text{inst}}^{(SW)}$

cf. $g_s^2 = -\epsilon_1 \epsilon_2$

\[\text{(end of page 22)}\]
where

$$Z_{\text{Nek}}(a_i, \epsilon_1, \epsilon_2, ; q) \equiv \sum_{k=0}^{\infty} q^k \int_{M_k} 1_{\epsilon_1, \epsilon_2, a_i}$$

acting as Gaussian cutoffs

computable (in mathematics)
by the localization technique
H. Nakajima

where $\epsilon_1$, $\epsilon_2$ acting as Gaussian cutoffs

Its form

$$Z_k = \sum_{|Y^{(1)}| + \cdots + |Y^{(N)}| = k} Z_{Y^{(1)}, \ldots, Y^{(N)}}$$

$Y^{(i)}$ : partition
\[ Z = \int d^N \lambda \left( \Delta(\lambda) \right)^{+2b_E^2} \exp \left( \frac{b_E}{g_s} \sum_{I=1}^{N} W(\lambda_I) \right) \]

\[ \Delta(\lambda) = \prod_{1 \leq I < J \leq N} (\lambda_I - \lambda_J) \]

\[ \text{generic for a while} \]

- Vir. constraints; insert \[ \sum_{I=1}^{N} \frac{\partial}{\partial \lambda_I} \frac{1}{z - \lambda_I} \] into

identify as \[ T(z) \bigg|_+ \], i.e. \[ \langle T(z) \bigg|_+ \rangle = 0 \]

- Introduce \[ J(z) = i \partial \phi(z) = \frac{1}{\sqrt{2g_s}} W'(z) + \sqrt{2b_E} \text{Tr} \frac{1}{z - M} \]

\[ T(z) = -\frac{1}{2} : \partial \phi(z)^2 : + \frac{iQ_E}{\sqrt{2}} \partial^2 \phi(z), \quad Q_E = b_E - \frac{1}{b_E} \]

and write as \[ \langle g_s^2 T(z) \rangle = \frac{1}{4} W'(z)^2 - \frac{Q_E}{2} g_s W''(z) - f(z) \]

where \[ f(z) \equiv \left\langle b_E g_s \sum_{I=1}^{N} \frac{W'(z) - W'(\lambda_I)}{z - \lambda_I} \right\rangle \]

- Separately define the curve \( (x, z) = (y(z), z) \) by

\[ \left\langle \left( x + \frac{ig_s}{\sqrt{2}} \partial \phi(z) \right) \left( x - \frac{ig_s}{\sqrt{2}} \partial \phi(z) \right) \right\rangle = x^2 - g_s^2 \langle T(z) \rangle = 0 \]

\[ [x, z] = Q_E g_s \]
An-1 quiver matrix model: 
constructed s.t. obeying $W_n$ constraints at finite $N_a$

$$Z \equiv \int \prod_{a=1}^{r} \left\{ \prod_{I=1}^{N_a} d\lambda^{(a)}_I \right\} \left( \Delta_{A_{n-1}}(\lambda) \right)^{b_E^2} \exp \left( \frac{b_E}{g_s} \sum_{a=1}^{r} \sum_{I=1}^{N_a} W_a(\lambda^{(a)}_I) \right)$$

$$\Delta_{A_{n-1}}(\lambda) = \prod_{a=1}^{r} \prod_{1 \leq I < J \leq N_a} (\lambda^{(a)}_I - \lambda^{(a)}_J)^2 \prod_{1 \leq a < b \leq r} \prod_{I=1}^{N_a} \prod_{J=1}^{N_b} (\lambda^{(a)}_I - \lambda^{(b)}_J)^{(\alpha_a, \alpha_b)}$$

- $\exists n$ spin 1 currents s.t. $\sum_{i=1}^{n} J_i(z) = 0$

$$J_i(z) = i\partial \varphi_i(z) = \frac{1}{t_i(z)} + b_E \sum_{a=1}^{n-1} (\delta_{i,a} - \delta_{i,a+1}) \text{Tr} \frac{1}{z - M_a}$$

$$t_i(z) = \sum_{a=i}^{n-1} W'_a(z) - \frac{1}{n} \sum_{a=1}^{n-1} a W'_a(z)$$

- $\det(x - ig_s \partial \phi(z)) := \prod_{1 \leq i < n} (x - g_s J_i(z))$ contains $W_n$ generators

- $W_n$ constraints $\left\langle \det(x - ig_s \partial \phi(z)) \right|_+ \right\rangle = 0$

- the curve $\sum \left\langle \det(x - ig_s \partial \phi(z)) \right\rangle = 0$

Isomorphism with the Witten-Gaiotto curve established in the planar limit this way.
• **The planar limit:** the singlet factorization and the curve factorizes as

\[ 0 = \prod_{i=1}^{n} (x - y_i(z)) \quad (x, z) = (y_i(z), z) \]

\[ \lim g_s J_i \]

• **3-Penner:** choose

\[ W_a(z) = \sum_{p=1}^{3} (\mu_p, \alpha_a) \log(q_p - z) \]

\[ q_0 = \infty, \quad q_1 = 0, \quad q_2 = 1, \quad q_3 = q \]

The curve is found to agree with the \((n, N_f = 2n)\) curve in Witten-Gaiotto form.
• **Isomorphism between matrix model curve and the Seiberg Witten curve for general** $n$

The spectral curve of the $A_{n-1}$ quiver matrix model has the form

$$x^n = \sum_{k=2}^{n} (-1)^{k-1} P_k(z)x^{n-k},$$

where

$$P_k(z) = \frac{Q_k(z)}{(z(z-1)(z-q))^k},$$

for some polynomials $Q_k(z)$ in $z$.

On the other hand, the Seiberg-Witten curve in the Gaiotto form:

$$x^n = \frac{P_{4}^{(2)}(t)x^{n-2}}{(t(t-1)(t-q_{UV}))^2} + \frac{P_{6}^{(3)}(t)x^{n-3}}{(t(t-1)(t-q_{UV}))^3} + \cdots + \frac{P_{2n}^{(n)}(t)}{(t(t-1)(t-q_{UV}))^n}$$

$$= \sum_{k=2}^{n} \frac{P_{2k}^{(k)}(t)}{(t(t-1)(t-q_{UV}))^k}x^{n-k},$$

where $P_{2k}^{(k)}(t)$ are degree $2k$ polynomials in $t$.

The two are evidently similar.
Matching of the residues of $y(z)dz \ (i = 1, \cdots, n)$ at $z = 1, q, 0, \infty$

and those of $xdt \ (i = 1, \cdots, n)$ at $t = 1, q_{UV}, 0, \infty$

$\Rightarrow$

$$
\mu_0 = \sum_{a=1}^{n-1} (-m_a + m_{a+1}) \Lambda^a, \quad \mu_1 = \sum_{a=1}^{n-1} (\tilde{m}_a - \tilde{m}_{a+1}) \Lambda^a,
$$

$$
\mu_2 = \left( \sum_{i=1}^{n} m_i \right) \Lambda^1, \quad \mu_3 = \left( \sum_{i=1}^{n} \tilde{m}_i \right) \Lambda^{n-1}.
$$

The matrix model potentials $W_a(z) \ (a = 1, 2, \ldots, n - 1)$ are fixed as

$$
W_a(z) = (\tilde{m}_a - \tilde{m}_{a+1}) \log z + \delta_{a,1} \left( \sum_{i=1}^{n} m_i \right) \log (1 - z)
$$

$$
+ \delta_{a,n-1} \left( \sum_{i=1}^{n} \tilde{m}_i \right) \log (q_{UV} - z).
$$
Sorry. This talk is patchwork.

warning: models hopping from one to the other

A tentative conclusion: \( \mathcal{F} = F_{\text{mat}} \)