5.3.2 Dirac zero modes

If $Y_a$ are block-diagonal, it is expected from the following picture that zero modes, namely, solutions to the first equation in (5.3.10) arise from off-diagonal blocks. Here, we note that zero modes in 6 dimensions yield Dirac zero modes in 4 dimensions. Diagonal blocks of $Y_a$ correspond to D-branes, and these D-branes can intersect each other. Open strings are stretched between intersecting D-branes. At an intersecting point, the string whose length is zero corresponds to the Dirac zero mode. The off-diagonal block of $\psi^{(6d)}$ corresponds to those open strings.

Assuming that $Y_a$ have the block-diagonal structure, we solve (5.1.13) for each block of $Y_a$ with a common $\zeta$. In the following analysis, we set $\zeta = 1$ without loss of generality. We concentrate on two of the diagonal blocks of $Y_a$ and the off-diagonal block of $\psi^{(6d)}$:

\begin{align*}
Y_a &= \begin{pmatrix} Y_a^{(1)} \\ Y_a^{(2)} \end{pmatrix}, \quad (5.3.11) \\
\psi^{(6d)} &= \begin{pmatrix} \varphi \end{pmatrix}, \quad (5.3.12)
\end{align*}

where the sizes of $Y_a^{(1)}$ and $Y_a^{(2)}$ are $N_{Y}^{(1)}$ and $N_{Y}^{(2)}$, respectively. Then, we solve the following eigenvalue problem:

\begin{equation}
\gamma_a^{\alpha\beta} \left[ Y_a^{(1)} \varphi_\beta - \varphi_\beta Y_a^{(2)} \right] = \lambda \varphi_\alpha, \quad (5.3.13)
\end{equation}

where $\varphi_\alpha$ ($\alpha = 1, \ldots, 8$) are eigenvectors, and eigenvalue $\lambda$ corresponds to a mass in the (3+1) dimensions. If $\lambda$ is eigenvalue of $\varphi_\alpha$, $-\lambda$ is also eigenvalue of $\varphi_\alpha$ from the second equation in (5.3.10). Here, note that $\lambda = 0$ corresponds to a Dirac zero mode. In order to see the picture of intersecting D-branes, we consider wave functions. $\varphi_\alpha$ obtained in our numerical calculation contains the left-handed and the right-handed ones, so we extract $\varphi_{L\alpha}$ and $\varphi_{R\alpha}$:

\begin{equation}
\varphi_{L\alpha} = \frac{1 - \gamma_2}{2} \varphi_\alpha, \quad \varphi_{R\alpha} = \frac{1 + \gamma_2}{2} \varphi_\alpha. \quad (5.3.14)
\end{equation}

Here, we choose $U' = 1_{N_X} \otimes U'$ with $U' \in SU(N_{Y})$ as $U$ in (2.3.4) such that SU($N$) transformation preserves the quasi-direct-product structure (5.1.3). Then, $M$, $Y_a$ and $\psi_\alpha^{(6d)}$ are transformed as follows:

\begin{equation}
M' = U' M U'^\dagger, \quad Y_a' = U' Y_a U'^\dagger, \quad \psi_\alpha^{(6d)} = U' \psi_\alpha^{(6d)} U'^\dagger. \quad (5.3.15)
\end{equation}

In particular, by restricting $U'$ to

\begin{equation}
U' = \begin{pmatrix} U'' & 0 \\ 0 & V'' \end{pmatrix}, \quad (5.3.16)
\end{equation}

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with $U'' \in \text{SU}(N_Y^{(1)})$ and $V'' \in \text{SU}(N_Y^{(2)})$, which keeps the structure (5.3.11), we transform the wave functions corresponding to the Dirac operator as

$$
\varphi_{L\alpha} \mapsto U'' \varphi_{L\alpha} V''^\dagger, \quad \varphi_{R\alpha} \mapsto U'' \varphi_{R\alpha} V''^\dagger.
$$

(5.3.17)

In the following sections, we use this transformation in order to see whether the wave function is localized.

### 5.3.3 Solving the Dirac equation in numerical calculation

We see from the statement below (5.3.13) that for each eigenvalue in (5.3.13) there is another eigenvalue with the same magnitude and the different sign. In this section, we concentrate on the lowest and the second lowest eigenvalues among $(8N_Y^{(1)}N_Y^{(2)}/2)$ positive eigenvalues\textsuperscript{21}. From the picture of the intersecting D-branes, we expect the emergence of Dirac zero modes when the two branes intersect at a point. This requires generically that the dimensionality of the branes adds up to 6. If the sum of the dimensionality is less than 6, the two branes do not intersect, and if the sum is more than 6, the two branes intersect but not at a point. In order to specify the dimensionality of the brane, say to $d$, we set $6 - d$ components of $Y_a$ to zero. In the following, we replace $a = 4, \ldots, 9$ with $a = 1, \ldots, 6$.

3d and 4d solutions are generated numerically with the algorithm described in section 5.1. On the other hand, the 2d solutions can be constructed analytically as follows. By putting

$$
\begin{bmatrix}
Y_1^{(1)}, Y_2^{(2)}
\end{bmatrix} = iZ,
$$

(5.3.18)

we reduce (5.1.13) with $\zeta = 1$ to

$$
\begin{bmatrix}
Y_2^{(1)}, Z
\end{bmatrix} = iY_1^{(1)}, \quad \begin{bmatrix}
Z, Y_1^{(1)}
\end{bmatrix} = iY_2^{(1)}.
$$

(5.3.19)

Here, (5.3.18) and (5.3.19) imply that $Y_a^{(1)}$ and $Z$ are generators of the SU(2) algebra $L_i$ ($i = 1, 2, 3$):

$$
Y_1^{(1)} = L_1, \quad Y_2^{(1)} = L_2, \quad Z = L_3.
$$

(5.3.20)

We use the irreducible representation for configurations of $Y_a^{(1)}$ without loss of generality because we can obtain spectra with the reducible representation from summation of those with the irreducible representations. The above construction suggests that the 2d brane is something like a “fuzzy disk”, which can be obtained by projecting a fuzzy sphere onto a

---

\textsuperscript{21}Note that $\varphi_\alpha$ is the element of $(8N_Y^{(1)}N_Y^{(2)})$-dimensional vector space and here we count the number of the eigenvalues including the degeneracy.
plane. More precisely, it should be regarded as two coinciding fuzzy disks corresponding to the two hemispheres of the fuzzy sphere.

We solve (5.3.13) for \( Y \) obtained numerically in this section. Except in the cases where 2d solutions are used, we obtain 4 solutions for each block, so we examine \( 4 \times 4 = 16 \) cases. In the cases where 2d solutions are used, we also examine 16 cases, which are given by one solution for \( Y^{(1)} \) and 16 solutions for \( Y^{(2)} \). For simplicity, we set \( N^{(1)}_Y = N^{(2)}_Y \) except in the 2d-4d ansatz.

**3d-3d ansatz**

We further make the following ansatz for \( Y^{(1)} \) and \( Y^{(2)} \). 3-dimensional manifolds intersect at a point in the 6-dimensional space, so we use 3d-3d ansatz, which implies that

\[
Y^{(1)}_1 \neq 0, 
Y^{(1)}_2 \neq 0, 
Y^{(1)}_3 \neq 0, 
Y^{(1)}_4 = Y^{(1)}_5 = Y^{(1)}_6 = 0, 
\]

\[
Y^{(2)}_1 = Y^{(2)}_2 = Y^{(2)}_3 = 0, 
Y^{(2)}_4 \neq 0, 
Y^{(2)}_5 \neq 0, 
Y^{(2)}_6 \neq 0. 
\] (5.3.21) (5.3.22)

Since there is the ambiguity of the scale (or normalization), we take the ratio of the average of the lowest eigenvalues, which are denoted by \( \mu_0 \), to that of the second lowest ones, which are denoted by \( \mu_1 \), for each \( N^{(1)}_Y \) in order to fix this ambiguity. In Fig. 5.6, we plot the ratios against \( 1/N^{(1)}_Y \) and fit them to the quadratic function of \( 1/N^{(1)}_Y \), \( s + t/N^{(1)}_Y + u/(N^{(1)}_Y)^2 \) with \( s = -0.04(7) \), \( t = 39(6) \), and \( u = -5(1) \times 10^2 \). The constant term, \( s \), converges to 0 in the \( N^{(1)}_Y \to \infty \) limit within error. Therefore, we can obtain Dirac zero modes in the \( N^{(1)}_Y \to \infty \) limit.

Next, we consider the wave function corresponding to the lowest eigenvalue for one of the 16 cases with \( N^{(1)}_Y = 64 \). Here, we choose \( U'' \) and \( V'' \) in (5.3.17) such that

\[
\varphi_{R1} \mapsto U'' \varphi_{R1} V'' 
\] (5.3.23)

becomes the singular value decomposition (SVD). Namely, \( \varphi_{R1} \) is a diagonal matrix, where the diagonal elements are the singular values. In Fig. 5.7, we plot \( |(\varphi_{R1})_{pq}|^2 \) and \( |(\varphi_{L5})_{pq}|^2 \) \((p, q = 1, \ldots, N^{(1)}_Y)\). The wave functions are localized at the (1,1) element, while other wave functions take almost 0. This is consistent with the picture that the right-handed zero mode is localized at a point where D-branes intersect.

**2d-4d ansatz**

We make 2d-4d ansatz in which \( Y^{(1)}_a \) and \( Y^{(2)}_a \) take the following configurations:

\[
Y^{(1)}_1 = L_1, 
Y^{(1)}_2 = L_2, 
Y^{(1)}_3 = Y^{(1)}_4 = Y^{(1)}_5 = Y^{(1)}_6 = 0. 
\] (5.3.24)
Figure 5.6: $\mu_0/\mu_1$ for $N_Y^{(1)} = 32, 40, 48, 56, \text{ and } 64$ in the 3d-3d ansatz are plotted against $1/N_Y^{(1)}$. The dashed curve is a fit of ratios to $s + t/N_Y^{(1)} + u/(N_Y^{(1)})^2$ with $s = -0.04(7)$, $t = 34(6)$, and $u = -5(1) \times 10^2$.

Figure 5.7: $|\langle \varphi_{RL} \rangle_{pq}|^2$ and $|\langle \varphi_{LS} \rangle_{pq}|^2$ ($p, q = 1, \ldots, N_Y^{(1)}$) for $N_Y^{(1)} = 64$ in the 3d-3d ansatz are plotted. The $(1, 1)$ element of them is non-zero and the other elements are almost zero. For other $\alpha$, all the elements are almost zero.
Figure 5.8: $\mu_0/\mu_1$ for $N_Y^{(1)} = 24, 28, 32, \text{and } 36$ with $N_Y^{(2)} = (N_Y^{(1)})^2/16$ in the 2d-4d ansatz are plotted against $1/N_Y^{(1)}$. The dashed curve is a fit of ratios to $s + t/N_Y^{(1)} + u/(N_Y^{(1)})^2$ with $s = -0.01(16)$, $t = 13(9)$, and $u = -1.3(1.3) \times 10^2$.

\begin{equation}
Y_1^{(2)} = Y_2^{(2)} = 0, \quad Y_3^{(2)} \neq 0, \quad Y_4^{(2)} \neq 0, \quad Y_5^{(2)} \neq 0, \quad Y_6^{(2)} \neq 0.
\end{equation}

(5.3.25)

For $Y_a^{(2)}$, we generate sixteen solutions obtained\textsuperscript{22} with $N_Y^{(2)} = (N_Y^{(1)})^2/16$.

In the present case, the eigenvalues turn out to have two-fold degeneracy, which may be understood from the fact that the 2d brane is actually something like two coinciding fuzzy disks (see, (5.3.20) and the lines below.). We take the ratio between the average of the 32 lowest eigenvalues and that of the 32 second lowest ones. In Fig. 5.8, we plot the ratios against $1/N_Y^{(1)}$ for $N_Y^{(1)} = 24, 28, 32, \text{and } 36$. We see that ratio converge to 0 when $N_Y^{(1)}$ increases. Therefore, we can obtain Dirac zero modes in this ansatz.

We again consider the wave function corresponding to one of the 2 lowest eigenvalues\textsuperscript{23} for one of the 16 cases with $N_Y^{(1)} = 64$. Here, we choose $U''$ and $V''$ in (5.3.17) such that (5.3.23) becomes the SVD. In Fig. 5.9, we plot $|\langle \varphi_{R\alpha} \rangle_{pq}|^2$. For $\alpha = 1$, some elements are non-zero and other elements are almost zero. For other $\alpha$, all the elements are non-zero. We also plot $|\langle \varphi_{L\alpha} \rangle_{pq}|^2$. For $\alpha = 5$, some elements are non-zero and other elements are almost zero. For other $\alpha$, all the elements are non-zero. $|\langle \varphi_{R1} \rangle_{pq}|^2$ and $|\langle \varphi_{L5} \rangle_{pq}|^2$ are almost localized at the $(1, 1)$ element. These results are consistent with the picture that

\textsuperscript{22}The chosen matrix size $N_Y^{(2)}$ for the 4d brane is motivated from the fact that the degrees of freedom on a lattice with a linear extent $L$ grow as $L^2$ and $L^4$ for 2d and 4d cases, respectively. The factor of $1/16$ in $N_Y^{(2)} = (N_Y^{(1)})^2/16$ is introduced to avoid having too large $N_Y^{(2)}$ to perform explicit calculations.

\textsuperscript{23}The situation with the other lowest eigenvalue is qualitatively the same. The same comment applies also to the case with the 2d-3d ansatz below.

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Figure 5.9: $|\langle \varphi_{R1} \rangle_{pq} |^2$ and $|\langle \varphi_{L5} \rangle_{pq} |^2$ ($p = 1, \ldots, N_Y^{(1)}$, $q = 1, \ldots, N_Y^{(2)}$) for $N_Y^{(1)} = 36$ in the 2d-4d ansatz are plotted. For $|\langle \varphi_{R1} \rangle |^2$ and $|\langle \varphi_{L5} \rangle |^2$, $(p, p)$ elements ($p$ is less than 10) are non-zero and the other elements are almost zero. For other $\alpha$, all the elements are almost zero.

The left-handed and right-handed zero modes appear from the intersecting D-branes.

3d-4d ansatz

We make 3d-4d ansatz in which $Y_a^{(1)}$ and $Y_a^{(2)}$ take the following configurations:

$$
Y_1^{(1)} \neq 0 \ , \ Y_2^{(1)} \neq 0 \ , \ Y_3^{(1)} \neq 0 \ , \ Y_4^{(1)} = Y_5^{(1)} = Y_6^{(1)} = 0 \ , \\
Y_1^{(2)} = Y_2^{(2)} = 0 \ , \ Y_3^{(2)} \neq 0 \ , \ Y_4^{(2)} \neq 0 \ , \ Y_5^{(2)} \neq 0 \ , \ Y_6^{(2)} \neq 0 .
$$

We again obtain a lowest eigenvalue and a second lowest one in each of the 16 cases. For each $N_Y^{(1)}$, we take the ratio between the average of the 16 lowest eigenvalues and that of the 16 second lowest ones. In Fig. 5.10, we plot the ratios against $1/N_Y^{(1)}$ for $N_Y^{(1)} = 32$, 48, and 64. They do not converge to 0 in the $N_Y^{(1)} \to \infty$ limit, so we cannot obtain Dirac zero modes in this ansatz. We again consider the wave function corresponding to the lowest eigenvalue for one of the 16 cases with $N_Y^{(1)} = 64$. Here, we choose $U''$ and $V''$ in (5.3.17) such that (5.3.23) becomes the SVD.

In Fig. 5.11, we plot $|\langle \varphi_{R1} \rangle_{pq} |^2$ and $|\langle \varphi_{L5} \rangle_{pq} |^2$. We find that the wave functions are not localized but have a long tail along the diagonal line. These results are consistent with the picture that the intersection does not occur at a point. The zero modes do not appear, and the wave functions corresponding to the lowest eigenvalue do not localize. Similar behaviors are observed for the 4d-4d ansatz.

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Figure 5.10: $\mu_0/\mu_1$ for $N_Y^{(1)} = 32$, 48, and 64 in the 3d-4d ansatz are plotted against $1/N_Y^{(1)}$.

Figure 5.11: $|\langle \varphi_{R1} \rangle_{pq}|^2$ and $|\langle \varphi_{L5} \rangle_{pq}|^2$ ($p, q = 1, \ldots, N_Y^{(1)}$) for $N_Y^{(1)} = 64$ in the 3d-4d ansatz are plotted. A lot of diagonal elements of them are non-zero and the other elements are almost zero. For other $\alpha$, all the elements are almost zero.
Figure 5.12: $\mu_0/\mu_1$ for $N_Y^{(1)} = 32, 48, \text{and} 64$ in the 2d-3d ansatz are plotted against $1/N_Y^{(1)}$.

**2d-3d ansatz**

We make 2d-3d ansatz in which $Y_a^{(1)}$ and $Y_a^{(2)}$ take the following configurations:

$$Y_1^{(1)} \neq 0, \ Y_2^{(1)} \neq 0, \ Y_3^{(1)} = Y_4^{(1)} = Y_5^{(1)} = Y_6^{(1)} = 0, \quad (5.3.28)$$

$$Y_1^{(2)} = Y_2^{(2)} = Y_3^{(2)} = 0, \ Y_4^{(2)} \neq 0, \ Y_5^{(2)} \neq 0, \ Y_6^{(2)} \neq 0. \quad (5.3.29)$$

We obtain 2 lowest eigenvalues and 2 second lowest ones in each of the 16 cases as in the 2d-4d ansatz. We take the ratio between the average of the 32 lowest eigenvalues and that of the 32 second lowest ones. In Fig. 5.12, we plot the ratios against $1/N_Y^{(1)}$ for $N_Y^{(1)} = 32, 48, \text{and} 64$. We see that they do not converge to 0 in the $N_Y^{(1)} \to \infty$ limit, so that we cannot obtain Dirac zero modes in this ansatz.

We consider the wave function corresponding to one of the 2 lowest eigenvalues for one of the 16 cases with $N_Y^{(1)} = 64$. Here, we choose $U''$ and $V''$ in (5.3.17) such that (5.3.23) becomes the SVD. In Fig. 5.13, we plot $|\langle \varphi_{R\alpha} \rangle_{pq}|^2$. For $\alpha = 1, 2$, the $(1, 1)$ element is non-zero and the other elements are almost zero. For $\alpha = 5, 6$, the $(1, 2)$ element is non-zero and the other elements are almost zero. For other $\alpha$, all the elements are almost zero. We also plot $|\langle \varphi_{L\alpha} \rangle_{pq}|^2$. For $\alpha = 1, 2$, the $(1, 2)$ element is non-zero and the other elements are almost zero. For $\alpha = 5, 6$, the $(1, 1)$ element is non-zero and the other elements are almost zero. For other $\alpha$, all the elements are almost zero. Therefore, in this ansatz, we cannot see the picture that D-branes intersect.
Figure 5.13: $|\langle \varphi_{R\alpha} \rangle_{pq}|^2$ and $|\langle \varphi_{L\alpha} \rangle_{pq}|^2$ ($\alpha = 1, 2, 5, 6, \ p, q = 1, \ldots, N_{Y}^{(1)}$) for $N_{Y}^{(1)} = 64$ in the 2d-3d ansatz are plotted. For $|\langle \varphi_{R\alpha} \rangle|^2$ with $\alpha = 1, 2$, the (1, 1) element is non-zero and the other elements are almost zero. For $|\langle \varphi_{R\alpha} \rangle|^2$ with $\alpha = 5, 6$, the (1, 2) element is non-zero and the other elements are almost zero. For other $\alpha$, all the elements are almost zero. For $|\langle \varphi_{L\alpha} \rangle|^2$ with $\alpha = 1, 2$, the (1, 2) element is non-zero and the other elements are almost zero. For $|\langle \varphi_{L\alpha} \rangle|^2$ with $\alpha = 5, 6$, the (1, 1) element is non-zero and the other elements are almost zero. For other $\alpha$, all the elements are almost zero.
Chapter 6

Conclusion and outlook

In this thesis, we have studied the three topics concerning space-times emerging from matrix models. We have seen the mechanism of emergence of space-times from the matrix models and properties of those emergent space-times in order to establish matrix models completely as non-perturbative formulations of superstring theory. In the following, we summarize the three topics studied in this thesis and show their outlook. Then, we conclude this thesis comprehensively and show a perspective of our studies.

Renormalization in a scalar field theory on the fuzzy sphere

In chapter 3, we studied renormalization in a scalar field theory on the fuzzy sphere by Monte Carlo simulation. We identified the Berezin symbol constructed from the Bloch coherent state as the field. It was shown that the 2-point and 4-point correlation functions are made independent of the UV cutoff, or the matrix size up to the wave function renormalization by tuning the mass parameter or the coupling constant. This strongly suggests that the theory is renormalizable non-perturbatively in the ordinary sense and that the theory is universal up to a parameter fine-tuning.

We also examined the 2-point and 4-point correlation functions on the phase boundary beyond which the $\mathbb{Z}_2$ symmetry is spontaneously broken. We found that the 2-point and 4-point correlation functions at different points on the boundary agree up to the wave function renormalization. This implies that the critical theory is universal, which is consistent with the above universality in the disordered phase, because the phase boundary is obtained by a parameter fine-tuning. Furthermore, it was observed that the 2-point correlation functions behave as those in a CFT at short distances and deviate universally from those at long distances. The latter is considered to be due to the UV/IR anomaly.

The CFT observed at short distances seems to be different from the critical Ising model (the scaling dimension of the spin operator $\Delta = 0.125$), because the value of $u$ in (3.3.8) disagrees with $2\Delta$. This suggests that the universality classes of the scalar field theory on the fuzzy sphere totally differs from those of an ordinary field theory\textsuperscript{24}.

\textsuperscript{24}It should be noted that the scaling dimension we obtained, $\Delta \simeq 0.075 = 3/40$, coincides with that of
Indeed, it was reported in [15–19] that there exists a novel phase in the theory on the fuzzy sphere which is called the non-uniformly ordered phase, or the stripe phase [82,83]. We hope to elucidate the universality classes by studying renormalization in the whole phase diagram.

The large-$N$ volume independence on group manifolds

In chapter 4, we showed that a theory on a group manifold $G$ is equivalent to the corresponding theory on $G/H$ with $H$ a subgroup of $G$ in the large-$N$ limit. The degrees of freedom on $G$ are retrieved by the degrees of freedom of matrices in a consistent way with the dimensional reduction to $G/H$. An advantage of reduction to $G/H$ with a finite volume compared to reduction to a matrix model is that one does not need to introduce $k$ multiplicity and take the $k \to \infty$ limit to extract only planar contribution as in the latter case [35,36], since the UV cutoff $V'/v'$ plays the role of extracting planar contribution. While we showed the equivalence perturbatively, we can show it non-perturbatively based on the continuum Schwinger-Dyson equations as in [41], by assuming the stability of the background, which is a counterpart of the center symmetry.

An interesting application of the large-$N$ equivalence in dimensional reduction on group manifolds is that the SU(2$|4$) symmetric gauge theory on $R \times S^2$ is equivalent to $N = 4$ super Yang-Mills theory on $R \times S^3$ in the large-$N$ limit. (For another large-$N$ equivalence between these two theories, see [37,84].) Both of the theories have gravity duals, so that the above equivalence would be seen on the gravity side. It is interesting to search for gravity duals of other large-$N$ equivalences in dimensional reduction [85–89].

Classical solutions in the Lorentzian type IIB matrix model

In chapter 5, we solved classical equations of motion of the type IIB matrix model numerically because the classical approximation is expected to be valid since the action becomes large due to the cosmic expansion. We obtained classical solutions by assuming a quasi-direct-product structure (5.1.3). This structure is compatible with SO(3,1) symmetry, and favors a block-diagonal structure which can yield D-branes intersecting in extra dimensions. We examined a space-time structure in (3+1) dimensions and Dirac zero modes in extra dimensions.

First, we focused on the (3+1)-dimensional space-time structure, which is represented by $X_\mu$ in (5.1.3). When $X_0$ is diagonalized, $X_i$ become band-diagonal, which ensures the locality of the time. We defined the time $t$ from eigenvalues of $X_0$ to derive the spin operator in the tricritical Ising model which is the (4,5) unitary minimal model.
time evolution of the space. Then, we found that the 3-dimensional space obtained in our solutions is smooth and SO(3) symmetry is respected at almost all of late time\textsuperscript{25}. Breaking of the SO(3) symmetry at the last regime is considered to be a finite matrix size effect, so it is expected to be disappeared in the \( N_X \to \infty \) limit. In [61], it was shown that the space-time structure is singular one which is represented by the Pauli matrices. This is attributed to approximation used in the simulation. From the results in [62] obtained by using the complex Langevin method, it is conjectured that the smooth space-time is obtained in the \( N \to \infty \) limit. Our results strongly support this conjecture.

Next, we examined Dirac zero modes in extra dimensions numerically by focusing on the structure in the extra dimensions. For simplicity, we concentrated on two of the blocks of \( Y_a \) in (5.1.3); \( Y_a^{(1)} \) and \( Y_a^{(2)} \). We further made ansatz for \( Y_a^{(1)} \) and \( Y_a^{(2)} \) in numerical solutions. In the 3d-3d and 2d-4d ansatz, we found solutions that give Dirac zero modes in the \( N_Y^{(1)} \to \infty \) limit. We fitted the ratios of the average of the lowest eigenvalues to that of the second lowest ones to the quadratic function in \( 1/N_Y^{(1)} \), and found that the ratio converges to 0 within error in the \( N_Y^{(1)} \to \infty \) limit. We also found that the wave functions corresponding to the lowest eigenvalues are localized at a point, which is consistent with the picture of intersecting D-branes. In other ansatz, we did not obtain Dirac zero modes and localized wave functions. What is important is that Dirac zero modes were obtained as solutions of classical equations of motion of the type IIB matrix model. In previous studies [69, 71, 73, 79], matrix configurations in extra dimensions are given by hand.

It is important to obtain solutions with larger \( N_X \) and see whether 3-dimensional space is completely uniform and SO(3) symmetric. Furthermore, from a viewpoint of cosmology, we would like to see whether the 3-dimensional space obtained in such solutions expands obeying a power law. We would also like to identify Higgs modes in the spectra of fluctuation of \( Y_a \), and determine the Yukawa coupling from the overlap of wave functions between Dirac zero modes and Higgs modes. By using the information of the Yukawa coupling and the renormalization group, we see whether chiral fermions are obtained at the low energy in (3+1) dimensions.

**Conclusion and perspective**

Throughout this thesis, we studied a mechanism of emergence of space-times from matrix models and properties of those emergent space-times. We showed that it is possible that field theories on non-commutative spaces describe the real world because renormalization is performed as in the ordinary field theories. We drew a lesson on the emergence of the

\textsuperscript{25}In [90], the emergence of 3-dimensional space obtained in our study was checked by using the same model.
curved space-times from the large-$N$ volume independence on group manifolds. We took a first step to verify a conjecture that the type IIB matrix model describes the real world by showing that one can obtain classical solutions which give rise to the (3+1)-dimensional expanding space-time and Dirac zero modes. It is expected that insights on field theories on non-commutative spaces and the curve space-times in matrix models can be used to verify the conjecture.

Superstring theory has not been established completely, and there remain a lot of aspects to be understood. In this thesis, we tried to understand some aspects through matrix models. We would like to not only develop the above studies but also study other aspects by using insights gained in this thesis. In particular, we would like to continue to study the type IIB matrix model. We hope to establish that superstring theory is indeed the unified theory including quantum gravity by obtaining both the Big-Bang universe and the Standard Model from the matrix model.

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Appendix A

Bloch coherent state and Berezin symbol

In this appendix, we summarize the basic properties of the Bloch coherent state \([76]\) and the Berezin symbol \([75]\).

The highest weight state \(|jj\rangle\)\(^{26}\) is considered to correspond to the north pole (see, Fig. A.1). Thus, the state \(|\Omega\rangle\) that corresponds to a point \(\Omega = (\theta, \varphi)\) is obtained by acting a rotation operator on \(|jj\rangle\):

\[
|\Omega\rangle = e^{i\theta (\sin \varphi L_1 - \cos \varphi L_2)} |jj\rangle . \tag{A.0.1}
\]

(A.0.1) implies that \(n_i L_i |\Omega\rangle = j |\Omega\rangle\), \((A.0.2)\)

where \(\vec{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)\). It is easy to show from (A.0.2) that \(|\Omega\rangle\) minimizes \(\sum_i (\Delta L_i)^2\) with \((\Delta L_i)^2\) being the standard deviation of \(L_i\).

It is convenient to introduce the stereographic projection, \(z = R \tan \frac{\theta}{2} e^{i\varphi}\). Then, (A.0.1) is rewritten as

\[
|\Omega\rangle = e^{z L_3/R} e^{-L_3 \log(1+|z/R|^2)} e^{-\bar{z} L_3/R} |jj\rangle , \tag{A.0.3}
\]

which gives an explicit form of \(|\Omega\rangle\) as

\[
|\Omega\rangle = \sum_{m=-j}^{j} \left( \frac{2j}{j+r} \right)^{1/2} (\cos \frac{\theta}{2})^{j+r} (\sin \frac{\theta}{2})^{j-r} e^{i(j-r)\varphi} |jr\rangle . \tag{A.0.4}
\]

By using (A.0.4), one can easily show the following relations:

\[
\langle \Omega_1 | \Omega_2 \rangle = \left[ \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} + e^{i(\varphi_2 - \varphi_1)} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right]^{2j} , \tag{A.0.5}
\]

\[
|\langle \Omega_1 | \Omega_2 \rangle|^2 = \left( \cos \frac{\chi}{2} \right)^{2j} \text{ with } \chi = \arccos(\vec{n}_1 \cdot \vec{n}_2) , \tag{A.0.6}
\]

\[
(2j + 1) \int \frac{d\Omega}{4\pi} |\Omega\rangle \langle \Omega| = 1 . \tag{A.0.7}
\]
(A.0.6) implies that the width of the Bloch coherent state is $R/\sqrt{j}$ for large $j$.

Denoting the Bloch coherent state $|\Omega\rangle$ by $|z\rangle$, we rewrite (A.0.4) and (A.0.7) as

$$|z\rangle = \left(\frac{z/R}{1 + |z/R|^2}\right)^j \sum_{r=-j}^{j} \left(\frac{2j}{j+r}\right)^{1/2} \left(\frac{R}{z}\right)^r |jr\rangle,$$

(A.0.8)

$$\frac{2j+1}{4\pi} 4R^2 \int \frac{d^2z}{(1 + |z/R|^2)^2} |z\rangle\langle z| = 1,$$

(A.0.9)

respectively.

The Berezin symbol for a matrix $\Phi$ with the matrix size $2j + 1$ is defined by

$$f_{\Phi}(\Omega) = f_{\Phi}(z, \bar{z}) = \langle \Omega | \Phi | \Omega \rangle = \langle z | \Phi | z \rangle.$$ 

(A.0.10)

By using (A.0.4), it is easy to show that

$$f_{[L_i, \Phi]}(\Omega) = L_i f_{\Phi}(\Omega).$$

(A.0.11)

(A.0.7) implies that

$$\frac{1}{N} \text{Tr}(\Phi) = \int \frac{d\Omega}{4\pi} f_{\Phi}(\Omega).$$

(A.0.12)

The definition of the star product for $\Phi$ and $\Phi'$ is

$$f_{\Phi} \ast f_{\Phi'}(\Omega) = f_{\Phi} \ast f_{\Phi'}(z, \bar{z}) = \langle \Omega | \Phi \Phi' | \Omega \rangle = \langle z | \Phi \Phi' | z \rangle.$$ 

(A.0.13)

Here let us consider a quantity

$$\frac{\langle w | \Phi | z \rangle}{\langle w | z \rangle},$$

(A.0.14)
which is holomorphic in $z$ and anti-holomorphic in $w$. Then, one can deform this quantity as follows:

$$
\frac{\langle w|\Phi|z \rangle}{\langle w|z \rangle} = e^{-\frac{\delta}{\pi} z} \frac{\langle w|\Phi|z + w \rangle}{\langle w|z + w \rangle} = e^{-\frac{\delta}{\pi} w} e^{z \frac{\delta}{\pi}} \frac{\langle w|\Phi|w \rangle}{\langle w|w \rangle} = e^{-\frac{\delta}{\pi} w} e^{z \frac{\delta}{\pi}} \langle w|\Phi|w \rangle = e^{-\frac{\delta}{\pi} w} e^{z \frac{\delta}{\pi}} f_{\Phi}(w, \bar{w}).
$$

(A.0.15)

Similarly, one obtains

$$
\frac{\langle z|\Phi|w \rangle}{\langle z|w \rangle} = e^{-\frac{\delta}{\pi} \bar{w}} e^{\bar{z} \frac{\delta}{\pi}} f_{\Phi}(w, \bar{w}).
$$

(A.0.16)

By using (A.0.9), (A.0.15) and (A.0.16), one can express the star product as

$$
f_{\Phi} \star f_{\Phi'}(w, \bar{w}) = \langle w|\Phi|w \rangle
\begin{align*}
&= \frac{2j + 1}{4\pi} \frac{4R^2}{(1 + |z/R|^2)^2} \left( e^{-\frac{\delta}{\pi} w} e^{z \frac{\delta}{\pi}} f_{\Phi}(w, \bar{w}) \right) \left( e^{-\frac{\delta}{\pi} \bar{w}} e^{\bar{z} \frac{\delta}{\pi}} f_{\Phi'}(w, \bar{w}) \right) \langle |w|z \rangle^2,
\end{align*}
$$

(A.0.17)

which indicates that the star product is non-commutative and non-local. Furthermore, one can easily show that in the $j \to \infty$ limit

$$
\frac{2j + 1}{4\pi} \frac{4R^2}{(1 + |z/R|^2)^2} \langle |w|z \rangle^2 \to \delta(2)(z - w).
$$

(A.0.18)

This implies that the star product coincides with the ordinary product in the $j \to \infty$ limit. Namely,

$$
f_{\Phi} \star f_{\Phi'}(w, \bar{w}) \to f_{\Phi}(w, \bar{w}) f_{\Phi'}(w, \bar{w})
$$

(A.0.19)

or

$$
f_{\Phi} \star f_{\Phi'}(\Omega) \to f_{\Phi}(\Omega) f_{\Phi'}(\Omega).
$$

(A.0.20)

We see from (A.0.11), (A.0.12) and (A.0.20) that the theory (3.1.1) reduces to the one (3.1.3) in the $N \to \infty$ limit at the classical level if one identifies $f_{\Phi}(\Omega)$ with $\phi(\Omega)$. However, the authors of [13,14] showed that the 1-loop effective action in (3.1.1) differs from that in (3.1.3) by finite and non-local terms since the UV cutoff $N$ is kept finite in calculating loop corrections. This phenomenon is sometimes called the UV/IR anomaly, which is explained in Appendix B.
Appendix B

UV/IR anomaly

There is no UV/IR mixing in a field theory on the fuzzy sphere. However, there exist finite differences between a theory on the fuzzy sphere and the corresponding theory on an ordinary sphere at the quantum level even in a continuum and commutative limit. These differences are the UV/IR anomaly [13]. In this appendix, we briefly review it.

From (2.1.60), $\Phi$ is expanded in terms of the fuzzy spherical harmonics $\hat{Y}^{[j]}_{lm}$:

$$\Phi = \sum_{l=0}^{2j} \sum_{m=-l}^{l} \phi_{lm} \hat{Y}^{[j]}_{lm}, \quad \phi^*_{lm} = (-1)^m \phi_{l-m}.$$  \hspace{1cm} (B.0.1)

$n$-point correlation functions are given by

$$\langle \phi_{l_1 m_1} \cdots \phi_{l_n m_n} \rangle = \frac{\int D\Phi e^{-S_{\text{fuzzy}} \phi_{l_1 m_1} \cdots \phi_{l_n m_n}}}{\int D\Phi e^{-S_{\text{fuzzy}}}},$$  \hspace{1cm} (B.0.2)

and in particular 2-point correlation functions take the form

$$\langle \phi_{lm} \phi^*_{l' m'} \rangle = (-1)^m \langle \phi_{lm} \phi_{l' -m'} \rangle = \frac{1}{l(l+1) + \mu^2 \delta_{ll'} \delta_{mm'}}.$$  \hspace{1cm} (B.0.3)

The vertex in the $\phi^4$ theory is given by

$$\phi_{l_1 m_1} \cdots \phi_{l_4 m_4} V(l_1, m_1; \ldots; l_4, m_4),$$  \hspace{1cm} (B.0.4)

where

$$V(l_1, m_1; \ldots; l_4, m_4) = \frac{\lambda}{4}(2j+1)(-1)^{l_1+l_2+l_3+l_4} \prod_{i=1}^{4} \sqrt{2l_i + 1} \sum_{l=0}^{2j} \sum_{m=-l}^{l} (-1)^m (2l + 1)$$

$$\times \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} l_3 & l_4 & l \\ m_3 & m_4 & -m \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l \\ j & j & j \end{pmatrix} \begin{pmatrix} l_3 & l_4 & l \\ j & j & j \end{pmatrix}$$  \hspace{1cm} (B.0.5)

is symmetric under cyclic permutations of $(l_i, m_i)$. Here, $\begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix}$ denotes 3j symbol.
Using (B.0.3), we calculate counter part of Fig. 2.1 and Fig. 2.2. The planar contribution \( \left( \Gamma_{\text{planar}}^{(2)} \right)_{m,m'}^{l,l'} \) is
\[
\left( \Gamma_{\text{planar}}^{(2)} \right)_{m,m'}^{l,l'} = 2\lambda \delta_{l,l'} \delta_{m,-m'} (-1)^m I^P,
\] (B.0.6)
\[I^P \equiv \sum_{K=0}^{2j} \frac{2K + 1}{K(K + 1) + \mu^2} \sim \log j + O(1).\] (B.0.7)

\( I^P \) exactly agrees with the corresponding term on the classical sphere in the \( j \to \infty \) limit.

The non-planar contribution \( \left( \Gamma_{\text{non-planar}}^{(2)} \right)_{m,m'}^{l,l'} \) is
\[
\left( \Gamma_{\text{non-planar}}^{(2)} \right)_{m,m'}^{l,l'} = \lambda \delta_{l,l'} \delta_{m,-m'} (-1)^m I^{NP},
\] (B.0.8)
\[I^{NP} \equiv \sum_{K=0}^{2j} (-1)^{l+K+2j} \frac{(2K + 1)(2j + 1)}{K(K + 1) + \mu^2} \left\{ \begin{array}{ccc} j & j & l \\ j & j & K \end{array} \right\}.\] (B.0.9)

We rewrite \( I^{NP} \) as follows:
\[I^{NP} = \sum_{K=0}^{2j} \frac{2K + 1}{K(K + 1) + \mu^2} f_K, \quad f_K \equiv (-1)^{l+K+2j}(2j + 1) \left\{ \begin{array}{ccc} j & j & l \\ j & j & K \end{array} \right\}.\] (B.0.10)

For \( l = 0 \), the planar contribution in the 2-point functions agree with the non-planar ones since \( f_K = 1 \) (\( 0 \leq K \leq 2j \)). For \( l = 1 \),
\[f_K = 1 - \frac{K(K + 1)}{2j(j + 1)},\] (B.0.11)
and then we obtain
\[I^{NP} = I^P - \frac{1}{2j(j + 1)} \sum_{K=0}^{2j} \frac{K(K + 1)(2K + 1)}{K(K + 1) + \mu^2}.\] (B.0.12)

Summation with respect to \( K \) in the second term of RHS in (B.0.12) diverges \( j^2 \), but thanks to \( 1/2j(j + 1) \), this term becomes finite. This implies that there is no IR singularity in the non-planar contribution unlike in the case of the non-commutative plane.

For general \( l \), we evaluate the non-planar contribution by using the approximation for the \( 6j \) symbol:
\[\left\{ \begin{array}{ccc} j & j & l \\ j & j & K \end{array} \right\} \approx (-1)^{l+2j+K} \frac{2j}{2j} P_l \left( 1 - \frac{K^2}{2j^2} \right)\] (B.0.13)
with \( l \ll j \) and \( 0 \leq K \leq 2j \). Here, \( P_l(x) \) is the Legendre polynomial\(^27\). Then, we obtain
\[I^{NP} - I^P = \sum_{K=0}^{2j} \frac{2K + 1}{K(K + 1) + \mu^2} \left( (-1)^{l+K+2j}(2j + 1) \left\{ \begin{array}{ccc} j & j & l \\ j & j & K \end{array} \right\} - 1 \right)\]
\[^{27}P_l(x) = \frac{1}{2l!} \frac{d^l}{dx^l} (x^2 - 1)^l.\]
\[
\approx \sum_{K=0}^{2j} \frac{2K+1}{K(K+1)+\mu^2} \left[ P_l \left(1 - \frac{K^2}{2j^2}\right) - 1 \right]. \tag{B.0.14}
\]

Because \(P_l(1) = 1\) for all \(l\), only \(K \gg 1\) contributes, and the summation is approximated by the integral:

\[
I^{NP} - I^P \approx \int_0^2 \frac{2u + \frac{1}{j}}{u^2 + \frac{u}{j} + \left(\frac{u}{j}\right)^2} \left[P_l \left(1 - \frac{u^2}{2}\right) - 1\right] = \int_{-1}^1 dt \frac{P_l(t) - 1}{1 - t} + O\left(\frac{1}{j}\right) \tag{B.0.15}
\]

with change of variables \(1 - u^2/2 = t\) and \(\mu \ll j\). One can show

\[
\int_{-1}^1 dt \frac{P_l(t) - 1}{1 - t} = -2 \sum_{k=1}^l \frac{1}{k} = -2h(l), \tag{B.0.16}
\]

where \(h(l)\) is the harmonic number and \(h(0) = 0\). \(h(l)\) is finite for small \(l\), while \(h(l) \approx \log l\) for large \(l\). Therefore, the 1-loop effective action is obtained as follows:

\[
S_{1\text{-loop}} = S_{\text{fuzzy}} + \frac{1}{2j+1} \text{Tr} \left[ \frac{1}{2} \Phi \left[ \delta \mu^2 - 2\lambda h\left(\tilde{\Delta}\right) \right] \Phi \right] + O\left(\frac{1}{j}\right), \tag{B.0.17}
\]

where

\[
\delta \mu^2 = 3\lambda \sum_{K=0}^{2j} \frac{2K+1}{K(K+1)+\mu^2} \tag{B.0.18}
\]

is the renormalized mass squared and \(\tilde{\Delta}\) is the function of the Laplacian whose eigenvalues are \(l\) on \(\hat{Y}_{lm}^{(j)}\). Thus, it turns out that a non-commutative effect is analytic in \(1/j\). This is a non-trivial finite quantum effect and has an \(l\)-dependence.

A field theory in the commutative limit is defined in the \(j \to \infty\) limit with fixed \(R\), \(\lambda\), and \(\mu^2\). In this limit, the 1-loop effective action on the fuzzy sphere differs from that on the commutative sphere by amount

\[
\Gamma_{\text{anomaly}}^{(2)} = -\frac{\lambda}{2j+1} \text{Tr} \left[ \Phi h(\tilde{\Delta}) \Phi \right]. \tag{B.0.19}
\]

This is non-local due to \(h(l) \approx \log l\) for large \(l\). This effect does not appear in an ordinary field theory, and is called the UV/IR anomaly.
Appendix C

3nj symbol

In this appendix, we present concrete expressions for the 3j and the 6j symbols [91].

The Clebsch-Gordan coefficient is represented as

\[ C_{l_1 m_1 l_2 - m_2}^{l' - m'} = (-1)^{l_1 - l_2 - m'} \sqrt{2l' + 1} \binom{l_1 & l_2 & l'}{m_1 & -m_2 & m'} \]  

(C.0.1)

in terms of the 3j symbol. From (C.0.1), one can show the following relations:

\[
\begin{align*}
\begin{pmatrix} l_3 & l_3 & l'' \\ m_3 & -m_3 & -m'' \end{pmatrix} &= \frac{(-1)^{l'' + m'' + l_3 - m_3}}{\sqrt{2l_3 + 1}} C_{l_3 - m_3}^{l'' m''} , \\
\begin{pmatrix} l & l' & 0 \\ m & m' & 0 \end{pmatrix} &= \frac{(-1)^{l - m}}{\sqrt{2l + 1}} \delta_{l,l'} \delta_{m,-m'} , \\
\begin{pmatrix} l & l_4 & l'' \\ -m & -m_4 & m'' \end{pmatrix} &= \frac{(-1)^{l'' + m'' - l_4 - m_4}}{\sqrt{2l_4 + 1}} C_{l'' m''}^{l_4 m_4} , \\
\begin{pmatrix} l'' & l_4 & l'' \\ m' & m_4 & -m'' \end{pmatrix} &= \frac{(-1)^{-l_4 + m'' - m'}}{\sqrt{2l_4 + 1}} C_{l'' m''}^{l_4 m_4} . 
\end{align*}
\]

(C.0.2) (C.0.3) (C.0.4) (C.0.5)

One can also show the following relations:

\[
\sum_{m_3} C_{l_3 - m_3}^{l'' m''} = (2l_3 + 1) \delta_{l'' 0} ,
\]

(C.0.6)

\[
\sum_{m_4, m''} C_{l'' m''}^{l_4 m_4} C_{l'' m''}^{l' m'} = \frac{2l_4 + 1}{2l + 1} \delta_{l_4 l'} \delta_{m_4 - m'} ,
\]

(C.0.7)

\[
\begin{align*}
\begin{pmatrix} l & l' & 0 \\ j & j & 0 \end{pmatrix} &= \frac{(-1)^{l + 2j}}{(2l + 1)(2j + 1)} \delta_{l,l'} , \\
\begin{pmatrix} l_3 & l_3 & 0 \\ j & j & 0 \end{pmatrix} &= \frac{(-1)^{l_3 + 2j}}{(2l_3 + 1)(2j + 1)} , \\
\sum_{l''} (-1)^{2j + l''} (2l'' + 1) \begin{pmatrix} l & l_4 & l'' \\ j & j & j \end{pmatrix} \begin{pmatrix} l & l_4 & l'' \\ j & j & j \end{pmatrix} = \begin{pmatrix} j & j & l \\
\end{pmatrix} . 
\end{align*}
\]

(C.0.8) (C.0.9) (C.0.10)
Appendix D

Method of a numerical simulation

In this appendix, we describe a numerical method and an error estimation used in chapter 3.

D.1 Hybrid Monte Carlo algorithm

To study renormalization on the fuzzy sphere, we use the Hybrid Monte Carlo (HMC) algorithm [92], which enables us to perform Monte Carlo simulations efficiently. Here, we briefly review the HMC algorithm.

We consider a system described by a set of dynamical variables denoted $\phi$. Suppose that the probability distribution is given by $e^{-S(\phi)}$. Then, the detailed balance is satisfied if a stochastic process is generated by the following procedure.

1. Take $\phi$ as an initial state, and generate the conjugate momentum $\pi$ with the following probability:

$$P(\pi) \propto e^{-\pi^2/2}.$$  \hfill (D.1.1)

2. Update $\phi$ and $\pi$ by using the molecular dynamics as follows:

$$\dot{\phi}(t) = \frac{\partial H[\phi(t), \pi(t)]}{\partial \pi(t)} = \pi(t),$$  \hfill (D.1.2)

$$\dot{\pi}(t) = -\frac{\partial H[\phi(t), \pi(t)]}{\partial \phi(t)} = -\frac{\partial S(\phi(t))}{\partial \phi(t)},$$  \hfill (D.1.3)

$$H[\phi(t), \pi(t)] = \frac{\pi^2}{2} + S(\phi(t)).$$  \hfill (D.1.4)

This expression is valid for the continuum time $t$ in the range $0 \leq t \leq \tau$. However, time must be discretized to perform numerical simulations discretely. We divide $\tau$ into $N_\tau$ steps. Here, we define $\varepsilon = \tau/N_\tau$. The equations of motion for molecular dynamics become difference equations. To discretize the time, we use a leap-frog method. In the following, $\phi(n)$ and $\pi(n)$ denotes $\phi(t = n\varepsilon)$ and $\pi(t = n\varepsilon)$, respectively.
Set initial conditions as follows:

\[ \phi(0) = 0, \quad \pi(0) = \pi, \quad (D.1.5) \]

then at the first step, take

\[ \pi\left(\frac{1}{2}\right) = \pi(0) - \varepsilon \frac{\partial S(\phi(0))}{\partial \phi(0)}. \quad (D.1.6) \]

At main steps \((n = 0, 1, \ldots, N-2)\), update \(\phi\) and \(\pi\):

\[ \phi(n+1) = \phi(n) + \varepsilon \pi \left(n + \frac{1}{2}\right), \quad \pi\left(n + \frac{3}{2}\right) = \pi\left(n + \frac{1}{2}\right) - \varepsilon \frac{\partial S(\phi(n+1))}{\partial \phi(n+1)}. \quad (D.1.7) \]

At the final step, \(\phi'\) and \(\pi'\), which are new \(\phi\) and \(\pi\), are

\[ \phi' = \phi(N-1) = \phi(N+1) - \varepsilon \pi \left(N - \frac{1}{2}\right), \quad (D.1.8) \]

\[ \pi' = \pi(N-1) = \pi\left(N - \frac{1}{2}\right) - \varepsilon \frac{\partial S(\phi(N+1))}{\partial \phi(N+1)}. \quad (D.1.9) \]

3. Accept \(\phi'\) and \(\pi'\) with the following probability, which is called the Metropolis test \cite{93}.

\[ P(\{\phi, \pi\} \rightarrow \{\phi', \pi'\}) = \min\{1, e^{-\Delta H}\} \quad \text{with} \quad \Delta H = H[\phi', \pi'] - H[\phi, \pi], \quad (D.1.10) \]

where

\[ \min\{1, e^{-\Delta H}\} = \begin{cases} 
\text{accept for } \Delta H < 0, \\
\text{accept with probability } e^{-\Delta H} \text{ for } \Delta H > 0.
\end{cases} \quad (D.1.11) \]

If \(\phi'\) and \(\pi'\) are rejected by the Metropolis test, \(\phi\) and \(\pi\) (old ones) are kept. In practice, we use a random number for the Metropolis test in numerical simulations.

4. Return to 1.

### D.2 Error estimation

Quantities calculated in numerical simulations have errors. In this appendix, we show how to estimate errors.

For observed quantities \(O_i\) \((i = 1, \ldots, N)\), their average and statistical error are given as follows:

\[ \langle O \rangle = \frac{1}{N} \sum_{i=1}^{N} O_i, \quad \delta \langle O \rangle = \sqrt{\frac{\langle O^2 \rangle - \langle O \rangle^2}{N-1}}. \quad (D.2.1) \]
For more complicated physical quantities, we use the following error-propagation relation:

$$\delta \langle f(\{O_a\}) \rangle = \sum_a \left| \left\langle \frac{\partial f}{\partial O_a} \right\rangle \delta \langle O_a \rangle \right|,$$

where $\{O_a\}$ denotes a set of physical quantities, and $f$ is an arbitrary function of them. However, in general, we excessively estimate an error for complicated quantities by using the above relation due to correlations among $O_a$.

We can overcome the problem by using the jackknife method. We show a procedure of the method for a bin size $n$.

1. Divide $N$ data into $N_n(\equiv N/n)$ equal parts. Each bin contains $n$ data.

2. Define the average except data in a bin $b$:

$$\langle O \rangle_b = \frac{1}{N-n} \sum_{k \notin B_b} O_k,$$

where $B_b$ denotes a set of indices for data in $b$.

3. By using $\langle O \rangle_b$, one can calculate the average and the error of $f$:

$$\langle f(O) \rangle = \frac{1}{N_n} \sum_{b=1}^{N_n} f(\langle O \rangle_b), \quad \delta \langle f(O) \rangle = \sqrt{N_n - 1} \left( \langle f(O)^2 \rangle - \langle f(O) \rangle^2 \right)^{1/2}.$$

(D.2.2)
Appendix E

Determination of the band size

We note here how we determine the band size $n$ for the classical solutions in section 5.2.2. Our model has a time reversal symmetry $t \rightarrow -t$, which comes from the invariance of the bosonic action $S_b$ under $X_0 \rightarrow -X_0$. Since $\Delta$ defined in (5.2.1) reflects this symmetry, it is almost symmetric under the exchange of the left-upper and the right-lower triangle:

$$\Delta_{pq} \rightarrow \Delta_{N_X+1-q, N_X+1-p}.$$  \hfill (E.0.1)

In order to determine the band size, we evaluate an average of $\Delta$ and its time reversal:

$$\bar{\Delta}_{pq} = \frac{1}{2} (\Delta_{pq} + \Delta_{N_X+1-q, N_X+1-p}).$$  \hfill (E.0.2)

$\bar{\Delta}$ in the typical solution for $N_X = 64$ is plotted against “a time separation” $\alpha_q - \alpha_p$ in Fig. E.1. Due to the (strict) monotonicity of the eigenvalues $\alpha_p$, the time separation is in one-to-one correspondence to “a distance” $q - p$ from the diagonal elements. In this plot, there are 20 series of points which correspond to $p + q = 4, 6, \ldots, 42$. We find that $\Delta$ scales only in a sufficiently large $\alpha_q - \alpha_p$ region. We determine the block size $n$ so that $\hat{X}_i(t)$ can cover a region where the scaling behavior is violated. In fact, this criterion is the same as the one adopted to determine $n$ in the previous studies [57–60]. From Fig. E.1, we determine $n = 10$ for the typical solution.
Figure E.1: $\Delta_{pq}$ are plotted against the time separation $\alpha_q - \alpha_p$. Each symbol corresponds to the label of “a slice of the band” $p+q-1 = 3, 5, \ldots, 41$. The scaling behavior is violated in the region $\alpha_q - \alpha_p \lesssim 0.15$
Appendix F

Detail calculations

In this appendix, we show detail calculations in this thesis.

F.1 Appendix A

(A.0.5) is calculated as follows:

\[
\langle \Omega_1 \mid \Omega_2 \rangle = \sum_{r=-j}^{j} \left( 2j \right) \left( \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^{j+r} \left( \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right)^{j-r} e^{i(j-r)(\phi_2-\phi_1)} \langle jr \mid jr \rangle
\]

\[
= \sum_{r=-j}^{j} \left( 2j \right) \left( \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^{j+r} \left( e^{i(\phi_2-\phi_1)} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right)^{j-r}
\]

\[
= \left[ \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\phi_2-\phi_1)} \right]^{2j} .
\] (F.1.1)

We consider the square of \( \langle \Omega_1 \mid \Omega_2 \rangle \):

\[
\left| \langle \Omega_1 \mid \Omega_2 \rangle \right|^2 = \left[ \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\phi_2-\phi_1)} \right]^{2j^2}
\]

\[
= \left[ \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + 2 \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\phi_2-\phi_1)} \right]^{2j}
\]

\[
= \left[ 1 + \cos \theta_1 \cos \theta_2 + \frac{1 - \cos \theta_1 \cos \theta_2}{2} + \frac{\sin \theta_1 \sin \theta_2}{4} + 2 \cos(\phi_2 - \phi_1) \right]^{2j}
\]

\[
= \left[ 1 + \cos \theta_1 \cos \theta_2 + \frac{\sin \theta_1 \sin \theta_2}{4} \right]^{2j} .
\] (F.1.2)

Thus, one obtains (A.0.6):

\[
\left| \langle \Omega_1 \mid \Omega_2 \rangle \right| = \left( \cos \frac{\chi}{2} \right)^{2j} .
\] (F.1.3)

Here, angle \( \chi \) is defined as follows:

\[
\vec{n}_1 \cdot \vec{n}_2 = (\sin \theta_1 \cos \varphi_1, \sin \theta_1 \sin \varphi_1, \cos \theta_1) \cdot (\sin \theta_2 \cos \varphi_2, \sin \theta_2 \sin \varphi_2, \cos \theta_2)
\]

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\[
\sin \theta_1 \cos \varphi_1 \sin \theta_2 \cos \varphi_2 + \sin \theta_1 \sin \varphi_1 \sin \theta_2 \sin \varphi_2 + \cos \theta_1 \cos \theta_2 \\
= \sin \theta_1 \sin \theta_2 (\cos \varphi_1 \cos \varphi_2 + \sin \varphi_1 \sin \varphi_2) + \cos \theta_1 \cos \theta_2 \\
= \sin \theta_1 \sin \theta_2 (\varphi_2 - \varphi_1) + \cos \theta_1 \cos \theta_2 \equiv \cos \chi .
\]

Here, we show (A.0.7):

\[ (2j + 1) \int_{-1}^{1} \frac{d\Omega}{4\pi} \langle \Omega | \Omega \rangle \\
= \frac{2j + 1}{4\pi} \int_{-1}^{1} d(\cos \theta) \int_{0}^{2\pi} \sin \theta \sin \varphi \left( \sum_{r,r'=-j}^{j} \left( \begin{array}{c} 2j \\ j + r \end{array} \right)^{\frac{1}{2}} \left( \begin{array}{c} 2j \\ j + r' \end{array} \right)^{\frac{1}{2}} \left( \cos \frac{\theta}{2} \right)^{2j+r+r'} \left( \sin \frac{\theta}{2} \right)^{2j-r-r'} e^{i(r-r')\varphi} |jr\rangle \langle jr'| \right) \\
= \frac{2j + 1}{2} \int_{-1}^{1} d(\cos \theta) \sum_{r=-j}^{j} \left( \begin{array}{c} 2j \\ j + r \end{array} \right) \left( \cos^{2} \frac{\theta}{2} \right)^{j+r} \left( \sin^{2} \frac{\theta}{2} \right)^{j-r} |jr\rangle \langle jr| \\
= \frac{2j + 1}{2^{2j+1}} \sum_{r=-j}^{j} \langle \Omega | (1 + x)^{2j} (-1)^{2j} d^2j \frac{d^2j}{dx^2} (1 - x)^{2j} |jr\rangle \langle jr| \\
= \frac{2j + 1}{2^{2j+1}} \sum_{r=-j}^{j} \langle \Omega | (1 + x)^{2j} |jr\rangle \langle jr| = \sum_{r=-j}^{j} |jr\rangle \langle jr| = 1 ,
\]

where we change \( \cos \theta \) to \( x \) and integrate by parts from (F.1.5) to (F.1.6).

Finally, we show (A.0.11). First of all, we derive \( f_{[L_3, \Phi]}(\Omega) \):

\[ f_{[L_3, \Phi]}(\Omega) = \langle \Omega | [L_3, \Phi] | \Omega \rangle = \langle \Omega | (L_3 \Phi - \Phi L_3) | \Omega \rangle \\
= \sum_{r,-r=-j}^{j} \left( \begin{array}{c} 2j \\ j + r \end{array} \right)^{\frac{1}{2}} \left( \begin{array}{c} 2j \\ j + r' \end{array} \right)^{\frac{1}{2}} \left( \cos \frac{\theta}{2} \right)^{2j+r+r'} \left( \sin \frac{\theta}{2} \right)^{2j-r-r'} e^{i(r-r')\varphi} \langle jr| \Phi | jr' \rangle \\
= -i \frac{\partial}{\partial \varphi} \langle \Omega | \Phi | \Omega \rangle = \mathcal{L}_3 f_{\Phi}(\Omega) .
\]

Next, \( f_{[L_+, \Phi]}(\Omega) \) is derived as

\[ f_{[L_+, \Phi]}(\Omega) = \langle \Omega | [L_+, \Phi] | \Omega \rangle \\
= \sum_{r,-r=-j}^{j} \left( \begin{array}{c} 2j \\ j + r \end{array} \right)^{\frac{1}{2}} \left( \begin{array}{c} 2j \\ j + r' \end{array} \right)^{\frac{1}{2}} \left( \cos \frac{\theta}{2} \right)^{2j+r+r'} \left( \sin \frac{\theta}{2} \right)^{2j-r-r'} e^{i(r-r')\varphi} \langle jr| (L_+ \Phi - \Phi L_+) | jr' \rangle
\]

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Here,
\[
\langle jr | L_+ = \sqrt{(j + r)(j - r + 1)} \langle j r - 1 | , \quad L_+ | j r' \rangle = \sqrt{(j - r')(j + r' + 1)} | j r' + 1 \rangle,
\]
so that
\[
(F.1.10) \quad (F.1.9) = \sum_{r,r'=-j}^{j} \left( \begin{array}{c} 2j \\ j + r \end{array} \right) \left( \begin{array}{c} 2j \\ j + r' \end{array} \right) \left( \cos \frac{\theta}{2} \right)^{2j+r+r'+1} \left( \sin \frac{\theta}{2} \right)^{2j-r-r'-1} \\
\times e^{i(r-r')\varphi} \sqrt{(j + r)(j - r + 1)} \langle j r - 1 | \Phi | j r' \rangle \\
- \sum_{r,r'=-j}^{j} \left( \begin{array}{c} 2j \\ j + r \end{array} \right) \left( \begin{array}{c} 2j \\ j + r' \end{array} \right) \left( \cos \frac{\theta}{2} \right)^{2j+r+r'-1} \left( \sin \frac{\theta}{2} \right)^{2j-r-r'+1} \\
\times e^{i(r-r')\varphi} \sqrt{(j + r')(j - r' + 1)} \langle jr | \Phi | jr' + 1 \rangle \\
= \sum_{r=-j}^{j-1} \sum_{r'=-j}^{j} \left( \begin{array}{c} 2j \\ j + r + 1 \end{array} \right) \left( \begin{array}{c} 2j \\ j + r' \end{array} \right) \left( \cos \frac{\theta}{2} \right)^{2j+r+r'+1} \left( \sin \frac{\theta}{2} \right)^{2j-r-r'-1} \\
\times e^{i(r-r')\varphi} \sqrt{(j + r)(j - r + 1)} \langle j r | \Phi | j r' \rangle \\
- \sum_{r=-j}^{j-1} \sum_{r'=-j}^{j+1} \left( \begin{array}{c} 2j \\ j + r \end{array} \right) \left( \begin{array}{c} 2j \\ j + r' - 1 \end{array} \right) \left( \cos \frac{\theta}{2} \right)^{2j+r+r'-1} \left( \sin \frac{\theta}{2} \right)^{2j-r-r'+1} \\
\times e^{i(r-r')\varphi} \sqrt{(j + r')(j - r' + 1)} \langle jr | \Phi | jr' \rangle \\
= \sum_{r=-j}^{j-1} \sum_{r'=-j}^{j} \left( \begin{array}{c} 2j \\ j + r \end{array} \right) \left( \begin{array}{c} 2j \\ j + r' \end{array} \right) \left( \cos \frac{\theta}{2} \right)^{2j+r+r'+1} \left( \sin \frac{\theta}{2} \right)^{2j-r-r'-1} \\
\times e^{i(r-r')\varphi} \sqrt{(j + r)(j - r + 1)} \langle j r | \Phi | j r' \rangle \\
- \sum_{r=-j}^{j-1} \sum_{r'=-j}^{j+1} \left( \begin{array}{c} 2j \\ j + r \end{array} \right) \left( \begin{array}{c} 2j \\ j + r' + 1 \end{array} \right) \left( \cos \frac{\theta}{2} \right)^{2j+r+r'+1} \left( \sin \frac{\theta}{2} \right)^{2j-r-r'-1} \\
\times e^{i(r-r')\varphi} \sqrt{(j + r')(j - r' + 1)} \langle jr | \Phi | jr' \rangle. 
\]

On the other hand, \( \mathcal{L}_+ \langle \Omega | \Phi | \Omega \rangle \) is
\[
\mathcal{L}_+ \langle \Omega | \Phi | \Omega \rangle \quad (F.1.12)
\]
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Thus, one obtains

\[ L_+ \langle \Omega | \Phi | \Omega \rangle = f_{[L_+ \Phi]}(\Omega) \]

Similarly, one can show \( L_- \langle \Omega | \Phi | \Omega \rangle = \)

\[ f_{[L_- \Phi]}(\Omega) \]

Thus, one obtains \( L_+ \langle \Omega | \Phi | \Omega \rangle = f_{[L_+ \Phi]}(\Omega) \). Similarly, one can show \( L_- \langle \Omega | \Phi | \Omega \rangle = \)

\[ f_{[L_- \Phi]}(\Omega) \]
First of all, let us consider
\[
\frac{\lambda}{4} \frac{1}{2j + 1} \text{Tr} \left( \Phi^4 \right) = \sum_{l_1, \ldots, l_4, m_1, \ldots, m_4} \phi_{l_1 m_1} \cdots \phi_{l_4 m_4} \frac{1}{4} \frac{1}{2j + 1} \text{Tr} \left( \hat{Y}_{l_1 m_1} \cdots \hat{Y}_{l_4 m_4} \right), \tag{F.2.1}
\]
and the fuzzy spherical harmonics \( \hat{Y}_l^{(j)} \) are abbreviated as \( \hat{Y}_{lm} \). The vertex (B.0.5) is calculated as
\[
V(l_1, m_1; \ldots; l_4, m_4) = \frac{\lambda}{4} \frac{1}{2j + 1} \text{Tr} \left( \hat{Y}_{l_1 m_1} \hat{Y}_{l_2 m_2} \hat{Y}_{l_3 m_3} \hat{Y}_{l_4 m_4} \right) = \frac{\lambda}{4} \sum_{l, l', m, m'} \tilde{C}_{l_1 m_1}^{l m} \tilde{C}_{l_2 m_2}^{l' m'} \frac{1}{2j + 1} \text{Tr} \left( \hat{Y}_{l m} \hat{Y}_{l' m'} \right)
= \frac{\lambda}{4} \sum_{l, l', m, m'} \tilde{C}_{l_1 m_1}^{l m} \tilde{C}_{l_2 m_2}^{l' m'} (-1)^m \delta_{l,l'} \delta_{m,m'}
= \frac{\lambda}{4} \sum_{l, l', m, m'} (-1)^m \tilde{C}_{l_1 m_1}^{l m} \tilde{C}_{l_2 m_2}^{l' m'}
= \frac{\lambda}{4} (2j + 1) \left\{ \sum_{i=1}^{4} \sqrt{2l_i + 1} \sum_{l', m'} (-1)^m \tilde{C}_{l_1 m_1}^{l m} \tilde{C}_{l_2 m_2}^{l' m'} C_{l_3 m_3}^{l m} C_{l_4 m_4}^{l' m'} \right\},
\]
where we use (2.1.46)–(2.1.49) and (C.0.1).

In Fig. F.1, we contract \( \phi_{l m} \) with \( \phi_{l_1 m_1}, \phi_{l_2 m_2} \) and \( \phi_{l_3 m_3} \) with \( \phi_{l_4 m_4} \). 8 possible contractions are the same contribution. (B.0.6) is obtained as follows:
\[
8 \sum_{l_1, \ldots, l_4, m_1, \ldots, m_4} (-1)^m \delta_{l_1, l'} \delta_{m_1, -m} \delta_{l_2, l'} \delta_{m_2, m} \frac{\lambda}{4} (2j + 1) (-1)^{\sum_{i=1}^{4} l_i} \frac{4}{\sqrt{2l_i + 1}},
\]

F.2 Appendix B
\[ \times \sum_{l''} (-1)^{m''}(2l'' + 1) \left( \begin{array}{ccc} l_1 & l_2 & l'' \\ m_1 & m_2 & m'' \end{array} \right) \left( \begin{array}{ccc} l_3 & l_4 & l'' \\ m_3 & m_4 & -m'' \end{array} \right) \{l_1\ l_2\ l''\} \{l_3\ l_4\ l''\} \{l_3\ l_4\ l''\} \{l_3\ l_4\ l''\} \]

\[ = 2\lambda(2j + 1) \frac{(-1)^m}{l(l + 1) + \mu^2} \sqrt{(2l + 1)(2l' + 1)} \sum_{l_3, m_3} \frac{(-1)^{-m_3}}{l_3(l_3 + 1) + \mu^2(-1)^{l''+l'+2l} (2l_3 + 1)} \]

\[ \times \sum_{l'', m''} \sum_{l', m'} \frac{2l_4 + 1}{l_5(l_5 + 1) + \mu^2} \{l\ l'\ l''\} \{l_3\ l_3\ l''\} \{l_3\ l_3\ l''\} \{l_3\ l_3\ l''\} \]

\[ = 2\lambda(2j + 1) \frac{(-1)^m}{l(l + 1) + \mu^2} \sqrt{(2l + 1)(2l' + 1)} \sum_{l_3, m_3} \frac{(-1)^{-m_3}}{l_3(l_3 + 1) + \mu^2(-1)^{l''+l'+2l} (2l_3 + 1)} \]

\[ \times \frac{2l_4 + 1}{l_5(l_5 + 1) + \mu^2} \{l\ l'\ l''\} \{l_3\ l_3\ l''\} \{l_3\ l_3\ l''\} \{l_3\ l_3\ l''\} \]

\[ = \left( \Gamma^{(2)}_{\text{planar}} \right)_{l''}^{l'} \times \frac{1}{l(l + 1) + \mu^2 l'(l' + 1) + \mu^2} , \]  

(F.2.3)

where \( l, l', l'', l_i, m, m', m'', m_i \) (\( i = 1, \ldots, 4 \)) are integers and \( j \) is half integer. To derive the above, we use (C.0.2)–(C.0.3), (C.0.8), and (C.0.9).

In Fig. F.2, we contract \( \phi_{lm} \) with \( \phi_{l_1m_1} \), \( \phi^*_{l'm'} \) with \( \phi_{l_3m_3} \), and \( \phi_{l_2m_2} \) with \( \phi_{l_4m_4} \). If one of \( \phi_{lm} \) (\( i = 1, \ldots, 4 \)) contracted with \( \phi_{lm} \) is chosen, another contracted with \( \phi^*_{l'm'} \) is
consequently chosen, so that there are 4 possible contractions to be the same contribution.

\[
4 \sum_{l_1, \ldots, l_4} \sum_{m_1, \ldots, m_4} (-1)^m \frac{\delta l_1 \delta_{m_1, -m}}{l_1(l_1 + 1) + \mu^2} \frac{\delta l_3 \delta_{m_3, -m_3}}{l_3(l_3 + 1) + \mu^2} = \]

\[
\times (-1)^{m_4} \frac{\delta l_2 \delta_{m_2, -m_4}}{l_2(l_2 + 1) + \mu^2} \frac{\lambda}{4} (2j + 1)(-1)^{\sum_{i=1}^4 l_i} \prod_{i=1}^4 \sqrt{2l_i + 1}
\]

\[
\times \sum_{m''} (-1)^{m''} (2l'' + 1) \left( \begin{array}{ccc} l_1 & l_2 & l'' \\ m_1 & m_2 & m'' \end{array} \right) \left( \begin{array}{ccc} l_3 & l_4 & l'' \\ m_3 & m_4 & -m'' \end{array} \right) \left\{ \begin{array}{ccc} l_1 & l_2 & l'' \\ j & j & j \end{array} \right\} \left\{ \begin{array}{ccc} l_3 & l_4 & l'' \\ j & j & j \end{array} \right\}
\]

\[
= \lambda(2j + 1) \frac{(-1)^m}{l(l + 1) + \mu^2} \frac{\sqrt{(2l + 1)(2l'' + 1)}}{|l'' + 1| + \mu^2} \sum_{l_1, m_1} (-1)^{-m_1} \frac{1}{l_1(l_1 + 1) + \mu^2} \frac{1}{2l_1 + 1} \frac{2l_1 + 1}{4} (2l + 1)(2l'' + 1)(2l_1 + 1)(2l_1 + 1)(2l_1 + 1)(2l_1 + 1)(2l_1 + 1) \]

\[
\times \sum_{m''} (-1)^{m''} (2l'' + 1) \left( \begin{array}{ccc} l & l_4 & l'' \\ -m & m_4 & m'' \end{array} \right) \left( \begin{array}{ccc} l & l_4 & l'' \\ m' & m_4 & -m'' \end{array} \right) \left\{ \begin{array}{ccc} l & l_4 & l'' \\ j & j & j \end{array} \right\} \left\{ \begin{array}{ccc} l & l_4 & l'' \\ j & j & j \end{array} \right\}
\]

\[
= \lambda(2j + 1) \frac{(-1)^m}{l(l + 1) + \mu^2} \frac{\sqrt{(2l + 1)(2l'' + 1)}}{|l'' + 1| + \mu^2} \sum_{l_1, m_1} \frac{2l_1 + 1}{4} (2l + 1)(2l'' + 1)(2l_1 + 1)(2l_1 + 1)(2l_1 + 1)(2l_1 + 1)(2l_1 + 1) \]

\[
\times \sum_{m''} (-1)^{2l + l'' + 3l'' + 3l_4 + m''} \frac{2l'' + 1}{2l_1 + 1} C_{l'' m''} C_{l'' m''} C_{l'' m''} C_{l'' m''} C_{l'' m''} C_{l'' m''} C_{l'' m''} C_{l'' m''} C_{l'' m''} \left\{ \begin{array}{ccc} l & l_4 & l'' \\ j & j & j \end{array} \right\} \left\{ \begin{array}{ccc} l & l_4 & l'' \\ j & j & j \end{array} \right\}
\]

\[
= \lambda(2j + 1) \frac{(-1)^m}{l(l + 1) + \mu^2} \frac{\sqrt{(2l + 1)(2l'' + 1)}}{|l'' + 1| + \mu^2} \sum_{l_1, m_1} \frac{2l_1 + 1}{4} (2l + 1)(2l'' + 1)(2l_1 + 1)(2l_1 + 1)(2l_1 + 1)(2l_1 + 1)(2l_1 + 1) \]

\[
\times \sum_{m''} (-1)^{2l + l'' + 3l'' + 3l_4 + m''} \frac{2l'' + 1}{2l_1 + 1} (2l'' + 1)(-1)^{3l + 2l''} \left\{ \begin{array}{ccc} l & l_4 & l'' \\ j & j & j \end{array} \right\} \left\{ \begin{array}{ccc} l & l_4 & l'' \\ j & j & j \end{array} \right\}
\]

\[
= \left[ \lambda(2j + 1) \delta_{l', l''} \delta_{m_1, -m_3} (-1)^m \sum_{l_1} \frac{(-1)^{l_1 + 2j}}{l_1(l_1 + 1) + \mu^2} \frac{4l_1 + 1}{l_1(l_1 + 1) + \mu^2} \frac{2l + 1}{2l_1 + 1} \right]
\]

Figure F.2: Contraction of non-planar diagram.
\[ \times \frac{1}{l(l+1) + \mu^2 (l'+1) + \mu^2} \frac{1}{\sqrt{2l'+1}} \]

\[ = \left( \Gamma_{\text{non-planar}}^{(2)} \right)_{m,m'}^{l,l'} \times \frac{1}{l(l+1) + \mu^2 (l'+1) + \mu^2} \frac{1}{\sqrt{2l'+1}}. \tag{F.2.4} \]

To derive the above, we use (C.0.4)–(C.0.7), and (C.0.10). (B.0.8) is derived in this way.
References


